18.03 Class 32, May 1

Eigenvalues and eigenvectors

[1] Prologue on Linear Algebra.

Recall [a b ; c d] [x ; y] = x[a ; c] + y[b ; d] :

A matrix times a column vector is the linear combination of the columns of the matrix weighted by the entries in the column vector.

When is this product zero?

One way is for x = y = 0. If [a ; c] and [b ; d] point in different directions, this is the ONLY way. But if they lie along a single line, we can find x and y so that the sum cancels.

Write A = [a b ; c d] and u = [x ; y] , so we have been thinking about A u = 0 as an equation in u . It always has the "trivial" solution u = 0 = [0 ; 0] : 0 is a linear combination of the two columns in a "trivial" way, with 0 coefficients, and we are asking when it is a linear combination of them in a different, "nontrivial" way.

We get a nonzero solution [x ; y] exactly when the slopes of the vectors
[a ; c] and [b ; d] coincide: c/a = d/b , or ad - bc = 0. This combination of the entries in A is so important it's called the "determinant" of the matrix:

det(A) = ad - bc

We have found:

Theorem: Au = 0 has a nontrivial solution exactly when det A = 0 .

If A is a larger *square* matrix the same theorem still holds, with the appropriate definition of the number det A .

[2] Solve u' = Au : for example with A = [1 2 ; 2 1] .

The "Linear Phase Portraits: Matrix Entry" Mathlet shows that some trajectories seem to be along straight lines. Let's find them first. That is to say, we are going to look for a solution of the form

u(t) = r(t) v

One thing for sure: the velocity vector u'(t) also points in the same (or reverse) direction as u(t). So for any vector v on this trajectory,

A v = lambda v

for some number lambda. This Greek letter is always used in this context.

[3] This is a pure linear algebra problem: A is a square matrix, and

we are looking for nonzero vectors v such that A v = lambda v for some number lambda. In order to get all the v's together, write the right hand side as

lambda v = (lambda I) v

where I is the identity matrix [1 0 ; 0 1] , and lambda I is the matrix with lambda down the diagonal. Then we can put this on the left:

0 = A v - (lambda I) v = (A - lambda I) v

Don't forget, we are looking for a nonzero v . We have just found an exact condition for such a solution:

det(A - lambda I) = 0

This is an equation in lambda ; we will find lambda first, and then set about solving for v (knowing in advance only that there IS a nonzero solution).

In our example, then, we subtract lambda from both diagonal entries and then take the determanent:

A - lambda I   = [1 - lambda , 2 ; 2 , 1 - lambda]

det (A - lambda I) = (1-lambda)(1-lambda) - 4

= 1 - 2 lambda + lambda^2 - 4

= lambda^2 - 2 lambda - 3

This is the "characteristic polynomial"

p_A(lambda) = det(A - lambda I)

of A , and its roots are the "characteristic values" or "eigenvalues" of A .

In our case, p_A(lambda) = (lambda + 1)(lambda - 3)

and there are two roots, lambda_1 = -1 and lambda_2 = 3 .

[4]   Now we can find those special directions. There is one line for lambda_1 and another for lambda_2 . We have to find nonzero solution v to

(A - lambda I) v = 0

eg with lambda = lambda_1 = -1 , A - lambda = [2 2 ; 2 2]

There is a nontrivial linear relation between the columns:

A [1 ; -1] = 0

All we are claiming is that

A [1 ; -1] = - [1 ; -1]

and you can check this directly. Any such v (even zero) is called an "eigenvector" of A.

Back to the differential equation. We have found that there is a straight line solution of the form r(t) v where   v = [1;-1] . We have

r' v = u' = A u = A rv = r A v = r lambda v

so (since v is nonzero)

r' = lambda r

and solving this goes straight back to Day One:

r = c e^{lambda t}

so for us   r = c e^{-t} and we have found our first straight line solution:

u = e^{-t} [1;-1]

In fact we've found all solutions which occur along that line:

u = c e^{-t} [1;-1]

Any one of these solutions is called a "normal mode."

General fact: the eigenvalue turns out to play a much more important role than it looked like it would: the straight line solutions are *exponential* solutions, e^{lambda t} v , where lambda is an eigenvalue for the matrix and v is a nonzero eigenvector for this eigenvalue.

The second eigenvalue, lambda_2 = 3 , leads to

A - lambda I = [-1 1 ; 1 -1]

and   [-1 1 ; 1 -1] v = 0   has nonzero solution v = [1;1]

so [1;1] is a nonzero eigenvector for the eigenvalue lambda = 3 , and there is another straight line solution

e^{3t} [1;1]

[5] The general solution to u' = Au will be a linear combination of the two eigensolutions (as long as there are two distinct eigenvalues).

In our example, the general solution is

u = c1 e^{-t} [1 ; -1] + c2 e^{3t} [1 ; 1]

We can solve for c1 and c2 using an initial condition: say for example u(0) = [2 ; 0]. Well,

u(0) = c1 [1 ; -1] + c2 [1 ; 1] = [c1+c2 ; -c1+c2]

and for this to be [2 ; 0] we must have c1 = c2 = 1:

u(t) = e^{-t} [1 ; -1] + e^{3t} [1 ; 1] .

When t is very negative, -10, say, the first term is very big and the second tiny: the solution is very near the line through [1 ; -1]. As t gets near zero, the two terms become comparable and the solution curves around. As t gets large, 10, say, the second term is very big and the first is tiny: the solution becomes asymptotic to the line through [1 ; 1].

The general solution is a combination of the two normal modes.

[6] Comments:

(1) The characteristic polynomial for the general 2x2 matrix A = [a,b;c,d] is

p_A(lambda) = (a-lambda)(d-lambda) - bc

= lambda^2 - (a+d) lambda - (ad-bc)

The sum of the diagonal terms of a square matrix is the "trace" of A , tr A, so

p_A(lambda) = lambda^2 - (tr A) lambda + (det A)

In our example, tr A = 2 and det A = 3 , and

p_A(lambda) = lambda^2 - 2 lambda - 3 .

(2) Any multiple of an eigenvector is another eigenvector for the same eigenvalue; they form a line, an "eigenline."

(3) The eigenlines for distinct eigenvalues are not genearally perpendicular to each other; that is a special feature of *symmetric* matrices, those for which b = c .

Also, generally the eigenvalues, roots of the characteristic polynomial, may be complex, not real. But for a symmetric matrix, all the eigenvalues are real.

Both these facts hold in higher dimensions as well. Most real numbers we know about are eigenvalues of symmetric matrices - the mass of an elementary particle, for example.