18.03 Class 33, May 3

Complex or repeated eigenvalues

[1] The method for solving u' = Au that we devised on Monday is this:

(1) Write down the characteristic polynomial

    p_A(lambda) = det(A - lambda I) = lambda^2 - (tr A)lambda +(det A)

(2) Find its roots, the eigenvalues   lambda_1, lambda_2

(3) For each eigenvalue find a nonzero eigenvector --- v such that

      Av = lambda v   or   (A - lambda I) v = 0

--- say v_1 , v_2.

Then the "ray" solutions are multiples of

      e^{lambda_1 t} v_1   and   e^{lambda_2 t} v_2

These are also called "normal modes." The general solution is a linear combination of them.

[2] This makes you think there are always ray solutions. But what about the Romeo and Juliet example, which spirals and obviously has no such solution? Or, what about

      A = [1 2 ; -2 1] .

I showed the trajectories on Linear Phase Portraits: Matrix Entry.

Let's apply the method and see what happens. tr(A) = 2 , det(A) = 5, so

      p_A(lambda) = lambda^2 - 2 lambda + 5

which has roots lambda_1 = 1 + 2i, lambda_2 = 1 - 2i.

(As always for real polynomials, the roots (if not real) come as complex conjugate pairs.)

We could abandon the effort at this point, but we had so much fun and success with complex numbers earlier that it seems we should carry on.

Find an eigenvector for lambda_1 = 1 + 2i :

      A - (1+2i)I : [- 2i , 2 ; -2 , -2i][? ; ?] = [0 ; 0]

Standard method: use the entries in the top row in reverse order with one sign changed: [2 ; 2i] or, easier, in this case,

      v_1   = [1 ; i].

This is set up so the top entry in the product is 0 . We have a chance to check our work (mainly the calculation of the eigenvalues) by seeing that the bottom entry in the product is 0 too:

$$-2 \cdot 1 - 2i \cdot i = 0$$

$[1 ; i]$ is a vector with complex entries. OK, so be it. It's hard to visualize, perhaps, and doesn't represent a point on the plane, but we can still compute with it just fine.

So one normal mode is

$$e^{(1+2i)t} [1 ; i]$$

[3] But we wanted real solutions. As in the case of second order equations,
the real and imaginary parts of solutions are again solutions:

So these are real solutions:

$$e^{(1+2i)t} [1 ; i]$$

$$= e^t (\cos(2t) + i \sin(2t)) ([1;0] + i[0;1]) \quad \text{so}$$

$$u_1 = Re(u) = e^t (\cos(2t) [1;0] - \sin(2t) [0;1])$$

$$= e^t [\cos(2t) ; - \sin(2t)] \quad \text{and}$$

$$u_2 = Im(u) = e^t (\cos(2t) [0;1] + \sin(2t) [1;0])$$

$$= e^t [\sin(2t) ; \cos(2t)]$$

These are two independent real solutions. Both spiral around the origin, clockwise, while fleeing away from it exponentially. They satisfy

$$u_1(0) = [1;0] \quad , \quad u_2(0) = [0;1] .$$

I showed their trajectories on the Mathlet Linear Phase Portraits: Matrix Entry.

The general real solution is

$$u = a u_1 + b u_2 \quad , \quad a, \ b \text{ real} .$$

It is very hard to visualize the fact that all
those spirals are linear combinations of any two of them.
In this case, it's easy to find a and b from $u(0)$:

$$u(0) = a[1;0] + b[0;1]$$

so $a = x(0)$ and $b = y(0)$. In general you have to solve a linear equation
to get a and b .

Note: Since λ_2 = conjugate of λ_1,
an eigenvector for λ_2 is given by the conjugate of v_1:

$$v_2 = [1 ; -i]$$

so another normal mode is e^{(1-2i)t} [1 ; -i] .
This is complex conjugate to the one we had before, so its real and imaginary parts give the same solutions we had before (up to sign).

Summary: Nonreal eigenvalues lead to spiral solutions.
Positive real parts lead to solutions going to infinity with t ("unstable")
Negative real parts lead to solutions going to zero with t ("stable")
Zero real parts lead to solutions parametrizing ellipses.

So we discover that the possibility of complex eigenvalues really isn't a failure of the method at all. There are in fact ray solutions, but they are complex and don't show up on our real phase plane.

[4] Second problem with our method: Illustrated by

A = [-2 1 ; -1 0]

p_A(lambda) = lambda^2 + 2 lambda + 1 = (lambda + 1)^2

which has only one root, a "repeated eigenvalue": lambda_1 = lambda_2 = -1.

Still, find an eigenvector:

A - (-1)I = [-1 1 ; -1 1][? ; ?] = [0 ; 0] : v_1 = [1 ; 1]

or any nonzero multiple. ALL eigenvctors for A lie on the line containing 0 and [1 ; 1]. I moved the sliders on Linear Phase Portraits: Matrix Entry to show the phase plane, which shows only one pair of opposed ray trajectories.

So there is (up to multiples) only one normal mode:

u_1 = e^{-t} [1 ; 1]

But we need another solution. Here is how to find one; I won't go into details, just give you the method.

Write down the same matrix A - lambda_1 I but now find a vector w such that

(A - lambda_1 I) w = v_1 .

Then

u_2 = e^{lambda_1 t} (t v_1 + w)

is a second solution.

In our case:

[-1 1 ; -1 1] [? ; ?] = [1 ; 1]

has solution [0;1] , so

u_2 = e^{-t} (t [1;1] + [0;1])

= e^{-t} [t ; t+1]

[0;1] isn't the only vector that works here; [0;1] + c v_1 does too for any constant c. It doesn't matter which one you pick.

With this choice, u_1(0) = [1;1] , u_2(0) = [0;1] .

The general solution is

u = a u_1 + b u_2 .

[5] A matrix with a repeated eigenvalue but only one lineful of eigenvectors is called "defective." A matrix can have a repeated eigenvalue and not be defective:

A = [2 0 ; 0 2]

for example has characteristic polynomial

p(lambda) = lambda^2 - 4 lambda + 4 = (lambda - 2)^2

so lambda_1 = lambda_2 = 2. To find an eigenvector consider

A - lambda_1 I : [0 0 ; 0 0] [? ; ?] = [0 ; 0]

Now ANY vector is an eigenvector! Instead of only one line you get the entire plane. For any vector v ,

e^{2t} v

is a solution, and every solution is a normal mode. This is called the "complete" case. Then you don't need the painful procedure described in

In the 2x2 case, if the eigenvalue is repeated you are in the defective case unless the matrix is precisely [lambda_1 , 0 ; 0 , lambda_1]

For larger square matrices this becomes the story of Jordan form.
To learn more about all this you should take 18.06 or 18/700.