18.03 Class 34, May 5

Classification of Linear Phase Portraits

The moral of today's lecture: Eigenvalues Rule (usually)

[1] Recall that the characteristic polynomial of a square matrix A is

p_A(lambda) = det(A - lambda I).

In the 2x2 case A = [a b ; c d] this can be rewritten as

p_A(lambda) = lambda^2 - (tr A) lambda + (det A)

where tr(A) = a + d , det(A) = ad - bc.

Its roots are the eigenvalues, so

p_A(lambda) = (lambda - lambda_1)(lambda - lambda_2)

= lambda^2 - (lambda_1 + lambda_2) lambda + (lambda_1 lambda_2)

Comparing coefficients,

tr(A) = lambda_1 + lambda_2 , det(A) = lambda_1 lambda_2

so the two numbers tr(A) and det(A) , extracted from the four numbers a, b, c, d, are determined by the eigenvalues. Conversely, they determine the eigenvalues, as the roots: by the quadratic formula,

lambda1,2 = tr(A)/2 +- sqrt(tr(A)^2/4 - det(A)).

lambda1,2 are not real if det(A) > tr(A)^2/4

are equal if det(A) = tr(A)^2/4

are real and different from each other if det(A) < tr(A)^2/4

The boundary is the "critical parabola," where det(A) = tr(A)^2/4.

Notice that if the eigenvalues are not real, their real part is tr(A)/2.
If the eigenvalues are real, they have the same sign exactly when their product is positive, and that sign is positive if their sum is also positive.

Relationship between tr and det vs eigenvalues

```
                          det

                           ^
     .                     |<-----purely imaginary
      .            complex roots              .
       .                   |                 .
        .     Re < 0       |     Re > 0     .<----- repeated
          .            |                 .
            .              |               .
   real < 0   .            |             .     real > 0
                ..         |          ..
                    ..     |      ..
  ------------        .......         ------------>  tr
                                            ^
                                            |
                  real, opposite sign       |______  at least one zero
 e.v.
```

The eigenvalues determine many general characteristics of the solutions.

[2] Spirals. We have seen that when the eigenvalues are non-real, we get spirals. Two more comments on this:

(1) The spirals move IN when Re(lambda) < 0 "stable spiral"
OUT when Re(lambda) > 0 "unstable spiral"

When Re(lambda) = 0 it turns out that the trajectories are ellipses. The technical term for this type of phase portrait is "center."

(2) We can tell which you get by thinking about what u' is at some point.
A convenient one to pick is [1;0] : then u' = Au = (first column of A).
So if the bottom left entry is positive, the spiral is moving counterclockwise;
if negative, it is moving clockwise.

[3] Saddles. When the eigenvalues are real and of opposite sign, the phase portrait is a "saddle." There are two eigenlines, one with positive
eigenvalue and the other with negative. Normal modes along one move out, and along the other move in. The general solution is a combination of these
two.

For example A = [1 2 ; 2 1] as we saw on Monday has phase portrait as shown on the Linear Phase Portraits: Cursor Entry, with tr = 2 , det = -3,
s = 0, and theta = 0 .

The eigenlines are of slope +-1 . The one of slope +1 corresponds to eigenvalue 3, the one of slope -1 has eigenvalue -1. If you take a solution
which is not a normal mode, when time gets large, the e^{-t} gets

small, and the e^{3t} is like the cube of $1/e^{-t}$: so the solution converges to the -1 eigenspace more quickly than it does to the +3 eigenspace.

By moving the trace slider you change the relative size of the eigenvalues; when tr = 0 they are of equal size.

Returning to tr = 2 , there are many matrices with this same pair of eigenvalues. The upper left box lets us explore them. One thing you can do is rotate the whole picture. The other thing you can do is change the angle between the eigenvectors.

[4] Nodes: When the eigenvalues are real and of the same sign, but distinct, you have a "node."

Eg A = [-2 , 1 ; 1 , -2] has

p_A(lambda) = lambda^2 + 4 lambda + 3 = (lambda + 1)(lambda + 3)

so the eigenvalues are -1 and -3. On the Mathlet (with tr = -4, det = 3, s = 0, theta = pi/2) you can see that the eigenlines are again of slope +- 1 : slope +1 with eigenvalue -1 and the other with eigenvalue -3. You can check this:

[-2 , 1 ; 1 , -2][1 ; 1] = [-1 ; -1]

[-2 , 1 ; 1 , -2][1 ; -1] = [-3 ; 3]

Both normal modes decay to zero, but the one with eigenvalue -3 decays much faster: so the non-normal mode trajectories become tangent to the eigenline with smaller |lambda|.

[5] The corresponding phase portraits exhibit the following behaviors:

```
                         det

                          ^
   .                      |<-----centers       .
    .                     |                   . <--- stars or
     .         stable     |   unstable       .       defective nodes
      .       spirals     |   spirals       .
        .             |              .
          .               |            .
  stable    .             |           .   unstable
  nodes       ..          |        ..     nodes
                  ..      |     ..
 ------------        .......        ------------>  tr
                                          ^
                                          |
                        saddles           |______  degenerate cases
```

There are also the special cases that happen along the curves separating these regions:

. det = 0 : "Degenerate." At least one of the eigenvalues is zero. If v is an eigenvector corresponding to this eigenvalue, then the constant vector valued function u(t) = c v is a solution for any constant c: there is a line (at least) of constant solutions. Several patterns are possible, and they are illustrated in the Supplementary Notes.

. det = tr^2/4 , along the critical parabola: repeated real eigenvalues.
The phase portraits are either
stars, in the complete case [lambda1 0 ; 0 lambda1] or
defective nodes , otherwise.

[6] Stability: All linear systems fall into one of the following categories:

Asymptotically stable: all solutions ---> 0 as t ---> infinity.
These systems occupy the upper left quadrant, tr < 0 and det > 0, so all the eigenvalues have negative real part.

Neutrally stable: all solutions stay bounded
but most don't ---> 0 as t --> infinity.
Ellipses and stable combs are examples.

Unstable: most solutions ---> infinity as t ---> infinity
Saddles and unstable nodes and spirals are examples.