18.03 Class 35, May 8

The companion matrix and its phase portrait;
The matrix exponential: initial value problems.

[1] We spent a lot of time studying the second order equation

$$x'' + bx' + kx = 0$$

and if b and k are nonnegative we interpreted them as the damping constant and spring constant (divided by the mass).

The companion system is obtained by setting

$$x' = y$$
$$y' = -kx - by$$

whose matrix of coefficients is the "companion matrix"

$$A = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix}$$

Note that $tr(A) = -b$, $det(A) = k$, and the characteristic polynomial of A is $\lambda^2 + b\lambda + k$, i.e., it is the same as the characteristic polynomial of the original second order equation.

If λ_1 is an eigenvalue of a companion matrix, then to find an eigenvector we look for v such that

$$\begin{bmatrix} -\lambda_1 & 1 \\ * & * \end{bmatrix} v = 0$$

$v = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$ does nicely. This makes sense: $x = e^{\lambda_1 t}$ has derivative $x' = \lambda_1 e^{\lambda_1 t}$, so

$$\begin{bmatrix} x \\ x' \end{bmatrix} = e^{\lambda t} \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$$

is a solution to the companion system.

QUESTION 1: What region in the (tr,det) plane corresponds to $c > 0$, $k > 0$?

Ans: the upper left quadrant.

QUESTION 2: What region in the (tr,det) plane corresponds to overdamping?

Ans: The part of the upper left quadrant which is below the critical parabola,
where there are stable nodes.

For example $x'' + (3/2)x' + (1/2)x = 0$ is overdamped. We saw long ago that
solutions to overdamped equations decay as time increases, can cross the $x = 0$ axis at most once, and can have at most one critical point (zero slope).

The companion matrix is $\begin{bmatrix} 0 & 1 \\ -1/2 & -3/2 \end{bmatrix}$. Roots of the characteristic polynomial, i.e. eigenvalues, are $-1/2$ and -1, with eigenvectors $[1 ; -1/2]$ and $[1 ; -1]$ respectively, so basic solutions

$$e^{-t/2} [1 ; -1/2] \quad \text{and} \quad e^{-t} [1 ; -1] .$$

There is a trajectory which is for very negative t roughly a line of slope -1 and large y intercept; it crosses the y axis at a point A, then hooks around, reaching an easterly most extreme point at B when it crosses the x axis, then reaches a southerly extreme at C, and finally heads in towards zero as $t \to \infty$.

What does the graph of the corresponding solution to $x'' + (3/2)x' + (1/2)x = 0$ look like? Where are the points on it corresponding to A, B, and C?

Answers: The solution is approximated by $c e^{-t}$ for $t << 0$, so its derivative is about $- c e^{-t}$. This explains why for $t << 0$ the slope of the trajectory is about -1. $x = 0$ at A ; the graph crosses the t axis. x reaches a maximum at B and then falls. The graph has a point of inflection where $y = x'$ has a minimum (and so $x'' = 0$), at C. It then decays exponentially to zero, well approximated by $c e^{-t/2}$ for $t >> 0$, so the derivative is about $(-c/2) e^{-t/2}$. This explains why for $t >> 0$ the slope of the trajectory is about $-1/2$.

Final reminder: in the companion system case, a stable spiral trajectory corresponds to a damped oscillation. Also improper nodes correspond to critically damped solutions.

[2] The Matrix Exponential

Recall from day one that $x' = ax$ with initial condition $x(0)$ has solution

$$x = x(0)e^{at} . \qquad (*)$$

This conveniently expresses the solution to this ODE with arbitrary initial condition (at $t = 0$). Also, e^{at} is DEFINED to be the solution to $x' = ax$ with initial condition $x(0) = 1$. This is what led us (and Euler before us) to the expression

$$e^{(a+bi)t} = e^{at} (\cos(bt) + i \sin(bt))$$

With this definition, (*) remains true.

Why not make an analogous definition in the case of a system of equations? That is, define the symbol e^{At} so that the solution to

$$u' = Au \text{ with initial condition } u(0) \text{ is } u(t) = e^{At}u(0). \qquad (**)$$

Note that the initial value $u(0)$ is a vector, and $u(t)$ is a vector valued function. So the expression e^{At} must denote a matrix, or rather a matrix valued function.

What could e^{At} be? What is its first column? Recall that the first column of any matrix B is the product $B[1;0]$. Combining this with (**) we see:

The first column of e^{At} is the solution to $u' = Au$ with $u(0) = [1;0]$.

Similarly,

The second column of e^{At} is the solution to $u' = Au$ with $u(0)= [0;1]$.

This is the DEFINITION of e^{At} . Note that at $t = 0$ we get the identity matrix. Let's check the claim (**) for any initial condition $[a;b]$. If we write v_1 and v_2 for those two solutions, so that

$$e^{At} = [v_1 , v_2] \quad \text{then}$$

$$e^{At} [a ; b] = a v_1 + b v_2$$

This is a solution (because it's a linear combination of solutions), and at $t = 0$ we get

$$a v_1(0) + b v_2(0) = a [1;0] + b [0;1] = [a;b] .$$

[3] We need a method of computing the e^{At}. Begin by finding a linearly independent pair of solutions (basic solutions, perhaps normal modes, as we have before). Call them u_1 and u_2. The general solution is a linear combination of u_1 and u_2, which we can write as

$$[u_1 , u_2] c_1$$

where c_1 is a column vector.

The matrix $[u_1 , u_2]$ is a "fundamental matrix" for A and is written $\Phi(t)$. The second column is also a solution :

$$\Phi(t) c_2$$

These two facts can be recorded using the matrix product

$$e^{At} = \Phi(t) [c_1 \quad c_2]$$

The right hand matrix is there to get the initial conditions right. Evaluate at $t = 0$:

$$I = \Phi(0) c$$

This means that $c = \Phi(0)^{-1}$.

Reminder on inverse matrices: A square matrix has an inverse exactly when its determinant is nonzero. In the 2x2 case A = [a,b;c,d] ,

[a b ; c d]^{-1} = (1/det A) [d -b ; -c a]

--- the diagonal terms get their positions reversed, and the off diagonal terms get their signs reversed.

Altogether: e^{At} = Phi(t) Phi(0)^{-1}

where Phi(t) is any fundamental matrix for A .

Example: A = [0 , 1 ; -1/2 , -3/2] .   We saw basic solutions

e^{-t/2} [1 ; -1/2]   and   e^{-t} [1 ; -1] .

so   Phi(t) = [e^{-t/2} , e^{-t} ; -(1/2)e^{-t/2} , -e^{-t}]

Phi(0) = [1 , 1 ; -1/2 , -1]

has determinant -1/2 and so

Phi(0)^{-1} = [2 , 2 ; -1 , -2] .

e^{At} = Phi(t) Phi(0)^{-1}

= [e^{-t/2} , e^{-t} ; -(1/2)e^{-t/2} , -e^{-t}] [2 , 2 ; -1 , -2]

= [2e^{-t/2}-e^{-t} , 2e^{-t/2}-2e^{-t} ; e^{-t/2}+e^{-t} , -e^{-t/2}+2e^t}]