18.03 Class 36, May 10

Review of matrix exponential
Inhomogeneous linear equations

[1] Prelude on linear algebra:

If A and B are matrices such that the number of columns in A is the same as the number of rows in B, then we can form the "product matrix" AB. If the columns of B are

$$b_1 , \ldots , b_n$$

then the columns of AB are

$$A b_1 , \ldots , A b_n$$

[2] I know you don't want to hear more about Romeo and Juliet. This is about amorous armadillos, named Xena and Yan.

$$x' = - x + 3y$$
$$y' = -3x - y$$

Matrix [-1 , 3 ; -3 , -1] .
Characteristic poly: lambda^2 + 2 lambda + 10
Eigenvalues: -1 +- 3i

Stop! what do we want to know? We have a stable spiral, rotating clockwise.
Do we want more? We already know that this romance will peter out into dull acceptance. Maybe that's enough information about X and Y's love life for us.

If not, we can go ahead and solve:
Eigenvector for -1 + 3i : [3 ; 3i] or just [1;i] .
Normal mode: e^{(-1+3i)t} [1;i]
Basic real solutions: e^{-t}[cos(3t) ; -sin(3t)]
e^{-t}[sin(3t) ; cos(3t)]

Fundamental matrix Phi(t) = e^{-t} [cos(3t) , sin(3t) ; -sin(3t) , cos(3t)]

Then the general solution is Phi(t) [a ; b] .

In fact you can think of Phi(t) as a matrix-valued solution:

Phi'(t) = A Phi(t) . (*)

By the linear algebra prologue, if B is any 2×2 matrix then the columns of Phi(t) B are of this form and hence are solutions.

The matrix exponential e^{At} is the fundamental matrix which is I when t = 0 . Its columns are the solutions which pass through [1;0] and

$[0;1]$
at $t = 0$. It must be of the form $\Phi(t) B$, and taking $t = 0$ shows that $\Phi(0) B = I$, or $B = \Phi(0)^{-1}$ and

$$e^{At} = \Phi(t) \Phi(0)^{-1}$$

For us, $\Phi(0) = I$ already! so we have found the matrix exponential.
The solution $u(t)$ with value $[a;b]$ at $t = 0$ is

$$e^{-t} \begin{bmatrix} \cos(3t) & \sin(3t) \\ -\sin(3t) & \cos(3t) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

[3] The matrix exponential has various familiar properties,

$$e^{As} e^{At} = e^{A(s+t)}$$

$$e^{A0} = I$$

$$(e^{At})^{-1} = e^{-At}$$

Evaluating it at t = other values gives other matrices with numbers (not functions) as entries; especially $t = 1$, which gives a definition of e^A.
Then the exponential law gives

$$e^{nA} = (e^A)^n \quad , \quad n \text{ any integer.}$$

Also, for those of you who love power series,

$$e^{A} = I + A + (A^2)/2 + (A^3)/3! + \dots.$$

Warning: $e^A e^B$ is generally different from e^{A+B}.

[4] Now, a well known perfume manufacturer has taken an interest in Xeno and Yan. They produce a scent, Armamour, which exerts a certain influence on X and on Y : it increases X's rate of change of affection by a and Y's by b. Now we have:

$$x' = - x + 3 y + a$$
$$y' = -3 x - y + b$$

This is an INHOMOGNENEOUS linear equation, which can be written

$$u' = A u + c$$

where $c = [a;b]$. Specifically, $c = [10;40]$.

Expect a constant solution u_p : then $u'_p = 0$, so

$$A u_p = - c \qquad \text{or} \qquad u_p = - A^{-1} c$$

With $A = [-1,3;-3,-1]$, $A^{-1} = (1/10)[-1,-3;3,-1]$, so

$$u_p = - [-.1,-.3;.3,-.1] c$$

Specifically, then,

u_p = [.1,.3;-.3,.1] [10;40] = [13;1] .

This is the "equilibrium solution."
The general solution is u_p + u_h where u_h is a solution of the homogeneous equation as above; it's a transient; all solutions spiral in to the single fixed point.

[5] Now, in fact, even Armani's armamour fades with time -- exponentially, of course. The truth is that q(t) = e^{-t}[10;40] .

Abstractly, we may have a forcing term that varies with time:

u' = Au + q(t)

We can solve this by "variation of parameters," as follows.

Let Phi(t) be a fundamental matrix for A . Instead of looking for solutions Phi(t) c , which work if q(t) = 0 , let's look for solutions of the form u = Phi(t) v(t) , where v(t) is as yet unknown.

(d/dt) Phi(t) v(t) = Phi'(t) v(t) + Phi(t) v'(t)

= A Phi(t) v(t) + q(t)

By (*), Phi'(t) = A Phi(t) , so those terms cancel and we find

Phi(t) v'(t) = q(t) or

v(t) = integral Phi(t)^{-1} q(t) dt so

u(t) = Phi(t) integral Phi(t)^{-1} q(t) dt

[6] In the Armani example, we have to work Phi(t)^{-1} . First,

det Phi(t) = e^{-2t}(cos^2(3t) + sin^2(3t)) = e^{-2t} so

Phi(t)^{-t} = e^{2t} e^{-t} [cos(3t) , -sin(3t) ; sin(3t) , cos(3t)]

= e^t [cos(3t) , -sin(3t) ; sin(3t) , cos(3t)]

Phi(t)^{-1} q(t) = [10 cos(3t) - 40 sin(3t) ; 10 sin(3t) + 40 cos(3t)]

integral Phi(t)^{-1} q(t) dt

= (10/3) [-sin(3t) - 4 cos(3t) ; cos(3t) - 4 sin(3t)] + c

u(t) = Phi(t) integral Phi(t)^{-t} q(t) dt

= (10/3) e^{-t} [-4 ; 1] + Phi(t) c

by an amazing cancellation which you should see for yourself.

The first term is the "particular solution," the second the homogeneous solution or transient. Despite Armani's best efforts, the relationship settles down to zip.