18.03 Muddy Card responses, May 10, 2006

1. "So what 'good' are exponential matrices? It seems to me that they don't allow us to skip any steps: it looks like you still have to calculate eigenvalues and then eigenvectors, and then use those calculations to construct the fundamental matrix and the inverse fundamental matrix at t=0. So how do they help us, other than providing a cute analogy to what we did earlier in the course?"

You know, the line between expressions of unity and coherence of a subject and cute analogies is a fuzzy one. The exponential function is a unifying theme of the course, and this is one more manifestation of it. I suggested that the expression $\mathbf{u}(t) = e^{At}\mathbf{u}(0)$ is convenient. A deeper reason to like the matrix exponential is that it lets you visualize the effect of the differential equation on the entire plane, all at once. Think of an unstable spiral: as time progresses, the points on a circle centered at the origin rotate and move closer to the origin. This is a "flow": you can think of it as fluid moving. In this case it is like water spiraling around a drain (or "sink," hence the name). The formula for this is this: at time t, the point $\mathbf{v}$ in the plane has been carried to $e^{At}\mathbf{v}$. When $t = 0$, things are where they start. This perspective makes it clear by the way that $e^{sA}e^{tA} = e^{(s+t)A}$.

2. What is the physical meaning of $\Phi(t)$? I hope what I said above clarifies that a little. In any case, $\Phi(t)$ is a "matrix-valued solution" to $\dot{\mathbf{u}} = A\mathbf{u}$: $\dot{\Phi}(t) = A\Phi(t)$. It is also required that neither column is a constant multiple of the other—they are "linearly independnent." The result is that the general solution has the nice expression $\Phi(t)\mathbf{c}$. I hope the calculations you do for homework clarify the meaning of $\Phi(t)$.

3. What is $e^A \begin{bmatrix} 1 \\ 0 \end{bmatrix}$? Answer: it is the solution to $\dot{\mathbf{u}} = A\mathbf{u}$ such that $u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

4. "When computing column vectors does the real part have to be c_1 and the imaginary be c_2? Or is it trivial? Why?" I'm pretty sure I don't understand the question. But let me say something about this real and imaginary part business. Suppose that A is a real 2×2 matrix whose eigenvalues are not real. They are then complex conjugates of each other, and the corresponding eigenvectors are complex conjugates of each other. So the two normal modes are complex conjugates of each other. Therefore their average is the real part of both, and the difference divided by $2i$ is the imaginary part of the first. This is one reason why the real and imaginary parts are again solutions: they are linear combinations of solutions. This observation also explains that we are not in fact forgetting about the second normal mode; it occurs as a term in both the real and imaginary parts of the first normal mode.