18.03 Class 37, May 12

Introduction to general nonlinear autonomous systems.

[1] Recall that an ODE is "autonomous" if x' depends only on x and not on t:

$$x' = g(x)$$

For example, I know an island in the St Lawrence River in upstate New York where there are a lot of deer. When there aren't many deer, they multiply with growth rate k ; $x' = kx$. Soon, though, they push up against the limitations of the island; the growth rate is a function of the population, and we might take it to be $k(1-(x/a))$ where a is the maximal sustainable population of deer on the island. So the equation is

$$x' = k(1-(x/a))x \quad , \quad \text{the "logistic equation."}$$

On this particular island, $k = 3$ and $a = 3$, so $x' = (3-x)x$.

There are "critical points" at $x = 0$ and $x = 3$. When $0 < x < 3$, $x' > 0$. When $x > 3$, $x' < 0$, and, unrealistically, when $x < 0$, $x' < 0$ too.

I drew some solutions, and then recalled the phase line:

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[2] One day, a wolf swims across from the neighboring island, pulls himself up the steep rocky shore, shakes the water off his fur, and sniffs the air. Two wolves, actually.

Wolves eat deer, and this has a depressing effect on the growth rate of deer. Let's model it by

$$x' = (3-x-y)\, x$$

where y measures the population of wolves.

Now, wolves in isolation follow a logistic equation too, say

$$y' = (1-y)\, y \qquad \text{(no deer)}$$

But the presence of deer increases their growth rate, say

$$y' = (1-y+x)\, y$$

We have a nonlinear autonomous system

$$x' = (3-x-y)\, x \qquad (*)$$

$y' = (1-y+x)\ y$

[3] The general case would be

$$x' = f(x,y)$$
$$y' = g(x,y)$$

-- the system does not change over time.

We have been studying a special case of this, the homogeneous linear case,
in which

$$f(x,y) = ax + by$$
$$g(x,y) = cx + dy$$

Giving an autonomous equation is the same thing as giving a vector field:

$$u'(t) = F(u) = f(x,y)\hat{i} + g(x,y)\hat{j}$$

I showed a slide of the linear vector field

$$F(u) = Au \quad \text{with} \quad A = [-1,3;-3,-1]$$

which modeled the effect of Armamour on Xena and Yan.

Solutions are parametrized curves such that the velocity vector at the point v is given by $F(v)$, both in direction and magnitude.
You can see some solutions, and how they thread their way through the vector field.

Then I showed a slide of the deer/wolf vector field. It seemed to show some "equilibria," points at which the vector field vanishes.

[4] The fact is that normally a process spends most of its time near equilibrium. Not exactly AT equilibrium, but near to it. It is always in the act of returning to equilibrium after being jarred off it by something.

So identifying equilibria is important, and understanding how the system behaves near equilibrium is important. To find the equilibria,
first think about where the vector field is vertical:

$$x' = 0 \text{ , that is } (3-x-y)\ x = 0 \text{ .}$$

This happens when either $x = 0$ or $3-x-y = 0$; the vector field is vertical along those two lines.

It is horizontal where

$$y' = 0 \text{ , that is } (1-y+x)\ y = 0 \text{ .}$$

This happens when either $y = 0$ or $1-y+x = 0$; the vector field is horizontal along those two lines.

The vector field is both vertical and horizontal exactly when it vanishes.

There are four places where that happens:

(0,0) , (3,0) , (0,1) ,

and the place where both $y = 3-x$ and $y = 1+x$, which is at (1,2) .

Notice that along the x axis, where $y = 0$, we get exactly the phase line
of the deer population without wolves, and along the y axis, where $x = 0$,
we get the phase line of the wolf population without deer. These two phase
lines sit inside the phase portrait of the deer/wolf system.

[5] We'll study the behavior near the critical point (0,0).

Expanding out,

$$x' = f(x,y) = 3x - x^2 - xy$$
$$y' = g(x,y) = y + xy - y^2$$

Near the origin the quadratic terms in the vector field are insignificant, so
"to first order"

$$x' = 3x$$
$$y' = y$$

The deer and wolf populations both expand by natural growth. This is a
homogeneous linear system with matrix [3 0 ; 0 1] . The eigenvalues
are 1 and 3 , so we have a node. The eigenline for the smaller eigenvalue
is the y axis, so this is the line to which all solutions (except for
the other normal mode!) all solutions become tangent as the approach the
origin.