18.03 Class 38, May 15

Nonlinear systems: Jacobian matrices

[1] The Nonlinear Pendulum.

The bob of a pendulum is attached to a rod, so it can swing clear around the pivot. This system is determined by three parameters:

- L length of pendulum
- m mass of bob
- g acceleration of gravity

We will assume that the motion is restricted to a plane.

To describe it we need a dynamical variable. We could use a horizontal displacement, but it turns out to be easier to write down the equation controlling it if you use the angle of displacement from straight down. Write theta for that angle, measured to the right.

Here is a force diagram:

```
      *
      |\
      |theta
      |  \
      |   \
   mg |   /\    (this is supposed to be a right angle!)
      |   \/
      |   /
      |  / mg sin(theta)
      | /
      |/
      *
```

Write s for arc length along the circle, with $s = 0$ straight down. Of course,

$$s = L\theta$$

Newton's law says

$$F = ms'' = mL\theta''$$

The force includes the $-mg\sin(\theta)$ component of the force of gravity (and notice the sign!), and also a frictional force which depends upon

$$s' = L\theta'$$

Friction is very nonlinear, in fact, but for the moment let's suppose that we are restricting to small enough values of θ' so that the behavior is linear. (It's surely zero when $\theta' = 0$.) So:

$$mL\theta'' = -mg\sin(\theta) - cL\theta'$$

Divide through by mL and we get

      theta" + b theta' + k sin(theta) = 0

where k = g/L and b = c/m .

This is a nonlinear second order equation. It still has a "companion first order system," obtained by setting

      x = theta   ,   y = x'

so y' = theta" = - k sin(theta) - b theta'   or

      x' = y
      y' = - k sin(x) - by

This is an autonomous system. Let's study its phase portrait.

[2] We studied the vector field for the deer/wolf population model, and so the differential equation, near (0,0)
by approximating it by a linear vector field. We can do that at any point.
We'll be particularly interested in approximating it near equilibria, say
near an equilibrium at (a,b) . Each coordinate in the vector field has a
tangent line approximation near (a,b) :

      f(x,y) ~ f(a,b) + f_x(a,b) (x-a) + f_y(a,b) (y-b)

Here f_x = partial f / partial x .

Since (a,b) is a critical point, the constant term vanishes and we have a
homogeneous linear system

      x' = f_x(a,b) (x-a) + f_y(a,b) (y-b)
      y' = g_x(a,b) (x-a) + g_y(a,b) (y-b)

With bar-x = x-a and bar-y = y-b this system
has matrix given by the "Jacobian matrix"

      J(x,y) = [f_x f_y ; g_x g_y]

evaluated at (a,b) . Near the critical point, the equation behaves like

      v' = J(a,b) v

where v = [bar-x ; bar-y] .

[3] In the pendulum, equilibria occur when also sin(x) = 0 .
This occurs when

      x = 0 , +-pi , +-2pi , ...

Let's compute the Jacobian:

J(x,y) = [0 , 1 ; - k cos(x) , -b]

When x = 0 , +-2pi , +-4pi , ..., cos(x) = 1 and

J(x,0) = [0 1 ; -k -b]

When x = +-pi , +-3pi , ..., cos(x) = -1 and

J(x,0) = [0 1 ; k -b]

Let's take some particular numbers: b = 2 , k = 65 .
In the first case

J(0,0) = [0 , 1 ; -65 , -2]

and the characteristic polynomial is

lambda^2 + 2 lambda + 65

with roots -1 +- 8i . The linear phase portrait is a spiral, moving counterclockwise. Solutions are of the form

x = A e^{-t} cos(8t - phi)

and of course y = x' . The period is 2pi/8 = pi/4 . In the course of one period, the magnitude decreases by a factor of e^{-pi/4} ~ .46 , or about half.

I sketched these at the appropriate critical points.

In the second case

J(pi,0) = [0 , 1 ; 65 , 2]

and the characteristic polynomial is

lambda^2 + 2 lambda - 65

with roots - 1 +- sqrt{66} ~ 7, -9

We have a saddle.

With eigenvalue near 7, A - lambda I has top row [- 7 1]
so the corresponding eigenvector is about [1 ; 7] .

Similarly, the eigenvector for eigenvalue -9 is about [1 ; -9] .

The eigenlines are both pretty steep, making a sharp V tilted somewhat to the right. I sketched these.

[4] Then I revealed the entire phase portrait. It shows several features
common to all companion systems:

- the trajectories cross the x axis perpendicularly.
- the trajectories above the x axis move right, those below move left.

Trajectories coming down from the left represent the pendulum swinging around in counterclockwise complete circles. Trajectories coming up from the right represent the pendulum swinging around in clockwise circles.

With a student I animated the pendulum swinging around. The successive dips represent passing through the vertical position. In very exceptional cases, the trajectory heads straight at the saddle equilibria; they converge to it like e^{-9t} , but most likely miss and move away like e^{7t} . The saddles represent the unstable equilibria which are straight up. Eventually, the trajectory gets caught in a basin (actually it was always in that basin) and spiral in towards the attractor of that basin, which is straight down. The spiral has period approaching 8 , and the amplitude of the swings decrease by about 50% with each swing.

[5] Postscript on the deer/wolf population model

$$x' = (3-x-y)\,x \qquad (*)$$
$$y' = (1-y+x)\,y$$

We found that the vector field is vertical where

$$x' = 0 \text{ , that is either } x = 0 \text{ or } y = 3 - x$$

and that the vector field is horizontal where

$$y' = 0 \text{ , that is either } y = 0 \text{ or } y = x + 1$$

Consequently there are critical points, or equilibria, at

$$(0,0) \text{ , } (3,0) \text{ , } (0,1) \text{ , } (1,2)$$

To analyze these, expand:

$$x' = f(x,y) = 3x - x^2 - xy$$
$$y' = g(x,y) = y + xy - y^2$$

Thus $J(x,y) = [\ 3 - 2x - y \ , -x \ ; \ y \ , \ 1 + x - 2y\ ]$

We evaluate this at the critical points.

For example, $J(0,0) = [\ 3 \ \ 0 \ ; \ 0 \ \ 1\ ]$ as we had before; an unstable node, with non-ray solutions becoming tangent to the y-axis as $t \to -\infty$.

We can do this with any of the critical points. For another example,

$$J(3,0) = [\ -3 \ \ -3 \ ; \ 0 \ \ 4\ ]$$

has eigenvalues -3 and 4 (since these are the diagonal entries in an upper triangular matrix ; so we have a saddle, with eigenvectors $[\ 1 \ ; \ 0\ ]$ and $[\ 3 \ ; \ -7\ ]$ respectively . These snap nicely into place.

Let's do the critical point at $(1,2)$ next:

$$J(1,2) = [\ -1 \ \ -1 \ ; \ 2 \ \ -2\ ]$$

has characteristic polynomial

$$\lambda^2 + 3\lambda + 4$$

and eigenvalues $-(3/2) \pm i\sqrt{7}/2$.

We have a stable spiral, rotating counterclockwise since the lower left entry is positive. I sketched this and snapped it into the nonlinear phase portrait.

These joined up to give an overall picture of the behavior of this system, the phase portrait.

We see that the entire upper right quadrant is a "basin," with the equilibrium (1,2) an "attractor."

We have learned that the populations of deer and of wolves converge to these stable levels no matter what the initial populations are (as long as they are positive). As the population levels approach this limiting value, they oscillate about them. We can say very exactly how this happens: the solutions for the deer look like

$$\bar{x} = e^{-(3/2)t} \cos((\sqrt{7}/2)t)$$

with exponential decay constant -3/2 and pseudofrequency $\sqrt{7}/2$ radians per unit time, or pseudoperiod $4\pi / \sqrt{7} \sim 4.75$ units of time.