18.03 Class 39, May 17

Dangers of linearization; limit cycles; chaos; next steps in mathematics

[1] The method we sketched on Monday works well "generically," i.e. almost all the time.

Things get tricky if the trace and determinant of the Jacobian matrix lie on one of the dividing curves in the (tr,det) plane.

If $tr = 0$ and $det > 0$, the linearization gives centers, ellipses, periodic trajectories. But the actual trajectories of the nonlinear system may not be periodic. They may be densely packed spirals. These spirals will rotate in the same direction as the predicted ellipse, but they may be either stable or unstable. It's as if the nonlinearity jostles the linear phase portrait off onto one of the regions bounded by the $tr = 0$ line.

Similarly, if $det = (tr/2)^2 > 0$, the linear phase plane is a star or a defective node. The actual phase plane near the equilibrium may be more like a spiral, or more like a node.

None of this is too bad, but there is one case where the actual phase portrait may not resemble the linear one at all. This is when the Jacobian matrix is degenerate, i.e. has determinant zero, i.e. has at least one eigenvalue which is zero.

For example,

$$x' = 4xy$$
$$y' = x^2 - y^2$$

$x' = 0$ if either $x = 0$ or $y = 0$. $y' = 0$ if $x = \pm y$.
So the only critical point is at $(0,0)$.

$$J(x,y) = [\ f_x , f_y ; g_x , g_y\ ] = [\ 4y , 4x ; 2x ; -2y\ ]$$

so $$J(0,0) = [\ 0\ 0 ; 0\ 0\ ]$$

What is the phase portrait of this linear system? It just says $u' = 0$ so every point is a constant solution. The eigenvalues are both zero, every vector is an eigenvector, it is a degenerate system.

On the other hand, I showed a Matlab plot of the phase portrait: it has six ray solutions, unlike any linear portrait. Weird.

We can still analyze this a bit. If $x = 0$ for example then and a solution occurs as long as $y' = -y^2$;
so there are other ray solutions along the vertical axis.

These solutions behave very differently from ray solutions of linear systems.

For example $y' = -y^2$ is separable: $y = 1/(t-c)$. This is two solutions, depending upon whether $t > c$ or $t < c$. This reflects the following behavior: Solutions for $y < 0$ start very slow but then speed up so quickly that they disappear to -infinity in finite time: graphed against time, they have a vertical asymptote. Solutions for $y > 0$ appear at some time, racing down from infinity, and then slow down as they near the origin. They approach the origin more slowly than any exponential decay (and the negative solutions begin their departure from the origin more slowly than any exponential growth).

The lesson: If you find $\det J(a,b) = 0$ you have to assume that the phase portrait near the critical point (a,b) will look different from the linear phase portrait. If $J(a,b)$ is nonzero, it will look the same.

[We can verify that there are ray solutions to this equation by finding the slope and substituting. If $y = mx$, then $x' = 4mx^2$ and $y' = (1-m^2)x^2$. The slope of the vector field must also be m, in order to keep the solution from wandering off the ray, so

$$m = y'/x' = (1-m^2)/4m \quad \text{or} \quad m = \pm 1/\sqrt{5}.$$

There are solutions along the lines through the origin with these slopes.]

[2] In studying a nonlinear autonomous system, the most important thing, after equilibria, is the possibility of periodic solutions.

For example, the "van der Pol" equation is

$$x'' + c\,(x^2-1)\,x' + x = 0$$

for fixed c. When $c = 0$ this is just the harmonic oscillator, with $\omega_n = 1$; all the nonzero solutions are periodic of period 2π.

The companion system is

$$x' = y$$
$$y' = -x - c\,(x^2-1)\,y$$

This system turns out to continue to have periodic solutions. When $c > 0$ the situation is in fact even better: there is ONLY ONE periodic trajectory, and all other nonzero solutions converge to it. This is a "limit cycle." The human heart (and I promise that this is the last time I'll mention it) is in fact controlled by an equation like this. This is why it returns to a normal periodic pattern after being disturbed.

[3] It can be shown that limit cycles are typical for 2D systems. But when you move to 3D things are much more complicated. The first such system was discovered right here at MIT by Edward Lorenz, who was modeling "convection rolls" in the upper atmosphere. In 1963 he wrote down a fairly simple model, a nonlinear autonomous system in 3 dimensions:

$$x' = -ax + by$$
$$y' = -xz + rx - y$$
$$z' = xy - bz$$

for constants a, b, r. Here's what happens for certain values of these parameters - and I showed the IDE tool "Lorenz Equations, Phase Plane, $0 < r < 30$. <http://www.aw-bc.com/ide/idefiles/media/JavaTools/lnrzphsp.html>.

The solutions don't ever settle down to a periodic orbit; but neither do they run off to infinity. They just wrap pretty crazily around the two nonzero unstable equilibria which exist provided that $r > a/b$.

[4] The unifying theme of the course has been the exponential function. See how many ways you can find it among the Ten Essential Skills:

1. First order models. Euler's method.

2. First order linear equations: Integrating factors or Variation of Parameter.

3. Complex numbers and exponentials.

4. Second order LTI systems, poly*exponential or sinusoidal signal; amplitude gain and phase lag.

5. Delta functions, unit impulse response, convolution.

6. Fourier series, periodic solutions.

7. Laplace transform; transfer function and weight function.

8. Linear systems: eigenvalues, eigenvectors; Variation of Parameters.

9. Linear phase portraits.

10. Nonlinear phase portraits; linearization at equilibria.

[5] Next steps in Mathematics

18.04 (spring) Complex Variables with Applications (poles and such; magical integral evaluations; Fourier analysis; conformal mappings)
(Alternatively, 18.112, requiring 18.100 as prerequisite.)

18.05 (spring) Probability and Statistics (Random variables; confidence intervals)
(Alternatively, 18.440, Probability, and 18.443, Statistics)

18.06 (fall and spring) Linear Algebra.

(Alternatively, 18.700, or 18.701)

18.100A (fall and spring), 18.100B (fall and spring), or 18.100C (spring):
Analysis I.

18.152 or 18.303: Linear PDEs, "pure" and "applied."

18.353J = 12.006 and 18.354J = 12.207: Nonlinear dynamics: Chaos, continuum systems; fluids, turbulence.....