18.03: Differential Equations, Spring, 2006

Driving through the dashpot

The Mathlet `Amplitude and Phase: Second order` considers a spring/mass/dashpot system driven through the spring. If $y(t)$ denotes the displacement of the plunger at the top of the spring, and $x(t)$ denotes the position of the mass, arranged so that $x = y$ when the spring is at rest, then we found the second order LTI equation

$$m\ddot{x} + b\dot{x} + kx = ky\,.$$

Now suppose instead that we fix the top of the spring and drive the system by moving the bottom of the dashpot instead. Here's a frequency response analysis of this problem. This time I'll keep m around, instead of setting it equal to 1 or dividing through by it. A new Mathlet, `Amplitude and Phase: Second Order, II`, illustrates this system with $m = 1$.

Suppose that the position of the bottom of the dashpot is given by $y(t)$, and again the mass is at $x(t)$, now arranged so that $x = 0$ when the spring is relaxed. Then the force on the mass is given by

$$m\ddot{x} = -kx + b\,\frac{d}{dt}(y - x)$$

since the force exerted by a dashpot is supposed to be proportional to the speed of the piston moving through it. This can be rewritten

$$m\ddot{x} + b\dot{x} + kx = b\dot{y}\,.$$

Suppose now that the motion is sinusoidal with circular frequency ω:

$$y = B\cos(\omega t)\,.$$

Then $\dot{y} = -\omega B\sin(\omega t)$ so our equation is

$$m\ddot{x} + b\dot{x} + kx = -b\omega B\sin(\omega t)\,.$$

Since $\mathrm{Re}\,(ie^{i\omega t}) = -\sin(\omega t)$, this equation is the real part of

$$m\ddot{z} + b\dot{z} + kz = bi\omega Be^{i\omega t}\,.$$

The Exponential Response Formula gives

$$z_p = \frac{bi\omega}{p(i\omega)}Be^{i\omega t}$$

where

$$p(s) = ms^2 + bs + k$$

is the characteristic polynomial. Thus the "transfer function" is

$$W(s) = \frac{bs}{p(s)}$$

and the complex gain is

$$W(i\omega) = \frac{bi\omega}{p(i\omega)}\,.$$

so that

$$z_p = W(i\omega)Be^{i\omega t}\,.$$

Using the natural frequency $\omega_n = \sqrt{k/m}$,

$$p(i\omega) = m(i\omega)^2 + bi\omega + m\omega_n^2 = m(\omega_n^2 - \omega^2) + bi\omega\,,$$

so

$$W(i\omega) = \frac{bi\omega}{m(\omega_n^2 - \omega^2) + bi\omega}\,.$$

We should certainly regard the "physical input signal" as $B\cos(\omega t)$, so I want to express the sinusoidal solution as

$$x_p = \text{gain} \cdot B\cos(\omega t - \phi)\,.$$

This is what we get with

$$\text{gain} = |W(i\omega)|$$

and

$$-\phi = \text{Arg}\,(W(i\omega))\,.$$

Thus both the gain and the phase are displayed by the curve parametrized by the complex valued function $W(i\omega)$. To understand this curve, divide numerator and denominator in the expression for $W(i\omega)$ by $bi\omega$:

$$W(i\omega) = \left(1 - \frac{i}{b/m}\,\frac{\omega_n^2 - \omega^2}{\omega}\right)^{-1}\,.$$

As ω goes from 0 to ∞, $(\omega_n^2 - \omega^2)/\omega$ goes from $+\infty$ to $-\infty$, so the expression inside the brackets follows the vertical straight line in the complex plane with real part 1, moving upwards. As z follows this line, $1/z$ follows a circle of radius $1/2$ and center $1/2$, traversed clockwise (exercise!). It crosses the real axis when $\omega = \omega_n$.

This circle is the "Nyquist plot." It shows that the gain starts small, grows to a maximum value of 1 exactly when $\omega = \omega_n$ (in contrast to the spring-driven situation, where the resonant peak is not exactly at ω_n and can be either very large or non-existent depending on the strength of the damping), and then falls back to zero. For large ω, $W(i\omega)$ is approximately $-ib/m\omega$, so the gain falls off like $(b/m)\omega^{-1}$.

The Nyquist plot also shows that $-\phi = \text{Arg}\,(W(i\omega))$ moves from near $\pi/2$ when ω is small, through 0 when $\omega = \omega_n$, to near $-\pi/2$ when ω is large.

And it shows that these two effects are linked to each other. Thus a narrow resonant peak corresponds to a rapid sweep across the far edge of the circle, which in turn corresponds to an abrupt phase transition from $-\phi$ near $\pi/2$ to $-\phi$ near $-\pi/2$.