Driving through the dashpot

The Mathlet Amplitude and Phase: Second order considers a spring/mass/dashpot system driven through the spring. If \( y(t) \) denotes the displacement of the plunger at the top of the spring, and \( x(t) \) denotes the position of the mass, arranged so that \( x = y \) when the spring is at rest, then we found the second order LTI equation

\[
m\ddot{x} + b\dot{x} + kx = ky.
\]

Now suppose instead that we fix the top of the spring and drive the system by moving the bottom of the dashpot instead. Here’s a frequency response analysis of this problem. This time I’ll keep \( m \) around, instead of setting it equal to 1 or dividing through by it. A new Mathlet, Amplitude and Phase: Second Order, II, illustrates this system with \( m = 1 \).

Suppose that the position of the bottom of the dashpot is given by \( y(t) \), and again the mass is at \( x(t) \), now arranged so that \( x = 0 \) when the spring is relaxed. Then the force on the mass is given by

\[
m\ddot{x} = -kx + b \frac{d}{dt}(y - x)
\]

since the force exerted by a dashpot is supposed to be proportional to the speed of the piston moving through it. This can be rewritten

\[
m\ddot{x} + b\dot{x} + kx = b\dot{y}.
\]

Suppose now that the motion is sinusoidal with circular frequency \( \omega \):

\[
y = B \cos(\omega t).
\]

Then \( \dot{y} = -\omega B \sin(\omega t) \) so our equation is

\[
m\ddot{x} + b\dot{x} + kx = -b\omega B \sin(\omega t).
\]

Since \( \text{Re}(ie^{i\omega t}) = -\sin(\omega t) \), this equation is the real part of

\[
m\ddot{z} + b\dot{z} + kz = bi\omega Be^{i\omega t}.
\]

The Exponential Response Formula gives

\[
z_p = \frac{bi\omega}{p(i\omega)} Be^{i\omega t}
\]

where

\[
p(s) = ms^2 + bs + k
\]

is the characteristic polynomial. Thus the “transfer function” is

\[
W(s) = \frac{bs}{p(s)}
\]
and the complex gain is
\[ W(i\omega) = \frac{b\omega}{p(i\omega)} . \]
so that
\[ z_p = W(i\omega)Be^{i\omega t} . \]
Using the natural frequency \( \omega_n = \sqrt{k/m} \),
\[ p(i\omega) = m(i\omega)^2 + bi\omega + m\omega^2_n = m(\omega^2_n - \omega^2) + bi\omega , \]
so
\[ W(i\omega) = \frac{bi\omega}{m(\omega^2_n - \omega^2) + bi\omega} . \]

We should certainly regard the “physical input signal” as \( B \cos(\omega t) \), so I want to express the sinusoidal solution as
\[ x_p = \text{gain} \cdot B \cos(\omega t - \phi) . \]
This is what we get with
\[ \text{gain} = |W(i\omega)| \]
and
\[ -\phi = \text{Arg} (W(i\omega)) . \]

Thus both the gain and the phase are displayed by the curve parametrized by the complex valued function \( W(i\omega) \). To understand this curve, divide numerator and denominator in the expression for \( W(i\omega) \) by \( bi\omega \):
\[ W(i\omega) = \left( 1 - \frac{i}{b/m\omega} \frac{\omega^2_n - \omega^2}{\omega} \right)^{-1} . \]
As \( \omega \) goes from 0 to \( \infty \), \( (\omega^2_n - \omega^2)/\omega \) goes from \( +\infty \) to \( -\infty \), so the expression inside the brackets follows the vertical straight line in the complex plane with real part 1, moving upwards. As \( z \) follows this line, \( 1/z \) follows a circle of radius 1/2 and center 1/2, traversed clockwise (exercise!). It crosses the real axis when \( \omega = \omega_n \).

This circle is the “Nyquist plot.” It shows that the gain starts small, grows to a maximum value of 1 exactly when \( \omega = \omega_n \) (in contrast to the spring-driven situation, where the resonant peak is not exactly at \( \omega_n \) and can be either very large or non-existent depending on the strength of the damping), and then falls back to zero. For large \( \omega \), \( W(i\omega) \) is approximately \(-ib/m\omega \), so the gain falls off like \((b/m)\omega^{-1}\).

The Nyquist plot also shows that \(-\phi = \text{Arg} (W(i\omega)) \) moves from near \( \pi/2 \) when \( \omega \) is small, through 0 when \( \omega = \omega_n \), to near \(-\pi/2 \) when \( \omega \) is large.

And it shows that these two effects are linked to each other. Thus a narrow resonant peak corresponds to a rapid sweep across the far edge of the circle, which in turn corresponds to an abrupt phase transition from \(-\phi \) near \( \pi/2 \) to \(-\phi \) near \(-\pi/2 \).