3. Laplace Transform

3A. Elementary Properties and Formulas

3A-1. Show from the definition of Laplace transform that $\mathcal{L}(t) = \frac{1}{s^2}$, $s > 0$.

3A-2. Derive the formulas for $\mathcal{L}(e^{at}\cos bt)$ and $\mathcal{L}(e^{at}\sin bt)$ by assuming the formula

$$\mathcal{L}(e^{\alpha t}) = \frac{1}{s-\alpha}$$

is also valid when α is a complex number; you will also need

$$\mathcal{L}(u+iv) = \mathcal{L}(u) + i\mathcal{L}(v),$$

for a complex-valued function $u(t) + iv(t)$.

3A-3. Find $\mathcal{L}^{-1}\big(F(s)\big)$ for each of the following, by using the Laplace transform formulas. (For (c) and (e) use a partial fractions decomposition.)

a) $\dfrac{1}{\frac{1}{2}s+3}$ b) $\dfrac{3}{s^2+4}$ c) $\dfrac{1}{s^2-4}$ d) $\dfrac{1+2s}{s^3}$ e) $\dfrac{1}{s^4-9s^2}$

3A-4. Deduce the formula for $\mathcal{L}(\sin at)$ from the definition of Laplace transform and the formula for $\mathcal{L}(\cos at)$, by using integration by parts.

3A-5. a) Find $\mathcal{L}(\cos^2 at)$ and $\mathcal{L}(\sin^2 at)$ by using a trigonometric identity to change the form of each of these functions.

b) Check your answers to part (a) by calculating $\mathcal{L}(\cos^2 at) + \mathcal{L}(\sin^2 at)$. By inspection, what should the answer be?

3A-6. a) Show that $\mathcal{L}\left(\frac{1}{\sqrt{t}}\right) = \sqrt{\frac{\pi}{s}}$, $s > 0$, by using the well-known integral

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}\,.$$

(Hint: Write down the definition of the Laplace transform, and make a change of variable in the integral to make it look like the one just given. Throughout this change of variable, s behaves like a constant.)

b) Deduce from the above formula that $\mathcal{L}(\sqrt{t}) = \frac{\sqrt{\pi}}{2s^{3/2}}$, $s > 0$.

3A-7. Prove that $\mathcal{L}(e^{t^2})$ does not exist for any interval of the form $s > a$.
(Show the definite integral does not converge for any value of s.)

3A-8. For what values of k will $\mathcal{L}(1/t^k)$ exist? (Write down the definition of this Laplace transform, and determine for what k it converges.)

3A-9. By using the table of formulas, find: a) $\mathcal{L}(e^{-t}\sin 3t)$ b) $\mathcal{L}(e^{2t}(t^2-3t+2))$

3A-10. Find $\mathcal{L}^{-1}\big(F(s)\big)$, if $F(s) =$

a) $\dfrac{3}{(s-2)^4}$ b) $\dfrac{1}{s(s-2)}$ c) $\dfrac{s+1}{s^2-4s+5}$

3B. Derivative Formulas; Solving ODE's

3B-1. Solve the following IVP's by using the Laplace transform:

a) $y' - y = e^{3t}, \quad y(0) = 1$ b) $y'' - 3y' + 2y = 0, \quad y(0) = 1,\ y'(0) = 1$

c) $y'' + 4y = \sin t, \quad y(0) = 1,\ y'(0) = 0$ d) $y'' - 2y' + 2y = 2e^t, \quad y(0) = 0,\ y'(0) = 1$

e) $y'' - 2y' + y = e^t, \quad y(0) = 1,\ y'(0) = 0$.

3B-2. Without referring to your book or to notes, derive the formula for $\mathcal{L}(f'(t))$ in terms of $\mathcal{L}(f(t))$. What are the assumptions on $f(t)$ and $f'(t)$?

3B-3. Find the Laplace transforms of the following, using formulas and tables:

a) $t\cos bt$ b) $t^n e^{kt}$ (two ways) c) $e^{at} t \sin t$

3B-4. Find $\mathcal{L}^{-1}(F(s))$ if $F(s) =$ a) $\dfrac{s}{(s^2+1)^2}$ b) $\dfrac{1}{(s^2+1)^2}$

3B-5. Without consulting your book or notes, derive the formulas

a) $\mathcal{L}\big(e^{at} f(t)\big) = F(s-a)$ b) $\mathcal{L}\big(t\, f(t)\big) = -F'(s)$

3B-6. If $y(t)$ is a solution to the IVP $y'' + ty = 0, \quad y(0) = 1,\ y'(0) = 0$, what ODE is satisfied by the function $Y(s) = \mathcal{L}(y(t))$?

(The solution $y(t)$ is called an *Airy function*; the ODE it satisfies is the *Airy equation*.)

3C. Discontinuous Functions

3C-1. Find the Laplace transforms of each of the following functions; do it as far as possible by expressing the functions in terms of known functions and using the tables, rather than by calculating from scratch. In each case, sketch the graph of $f(t)$. (Use the unit step function $u(t)$ wherever possible.)

a) $f(t) = \begin{cases} 1, & 0 \le t \le 1 \\ -1, & 1 < t \le 2 \\ 0, & \text{otherwise} \end{cases}$ b) $f(t) = \begin{cases} t, & 0 \le t \le 1 \\ 2-t, & 1 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$

c) $f(t) = |\sin t|, \quad t \ge 0.$

3C-2. Find $\mathcal{L}^{-1}$ for the following: a) $\dfrac{e^{-s}}{s^2+3s+2}$ b) $\dfrac{e^{-s} - e^{-3s}}{s}$ (sketch answer)

3C-3. Find $\mathcal{L}(f(t))$ for the square wave $f(t) = \begin{cases} 1, & 2n \le t \le 2n+1,\ n = 0,1,2,\dots \\ 0, & \text{otherwise} \end{cases}$

a) directly from the definition of Laplace transform;

b) by expressing $f(t)$ as the sum of an infinite series of functions, taking the Laplace transform of the series term-by-term, and then adding up the infinite series of Laplace transforms.

3C-4. Solve by the Laplace transform the following IVP, where $h(t) = \begin{cases} 1, & \pi \le t \le 2\pi, \\ 0, & \text{otherwise} \end{cases}$

$$y'' + 2y' + 2y = h(t), \quad y(0) = 0,\ y'(0) = 1;$$

write the solution in the format used for $h(t)$.

3C-5. Solve the IVP: $y'' - 3y' + 2y = r(t), \quad y(0) = 1,\ y'(0) = 0,$ where $r(t) = u(t)\,t$, the ramp function.

3D. Convolution and Delta Function

3D-1. Solve the IVP: $y'' + 2y' + y = \delta(t) + u(t-1), \quad y(0) = 0,\ y'(0^-) = 1.$

Write the answer in the "cases" format $y(t) = \begin{cases} \cdots, & 0 \le t \le 1 \\ \cdots, & t > 1 \end{cases}$

3D-2. Solve the IVP: $y''+y = r(t), \quad y(0) = 0,\ y'(0) = 1,$ where $r(t) = \begin{cases} 1, & 0 \le x \le \pi \\ 0, & \text{otherwise.} \end{cases}$

Write the answer in the "cases" format (see 3D-1 above).

3D-3. If $f(t+c) = f(t)$ for all t, where c is a fixed positive constant, the function $f(t)$ is said to be *periodic*, with period c. (For example, $\sin x$ is periodic, with period 2π.)

a) Show that if $f(t)$ is periodic with period c, then its Laplace transform is

$$F(s) = \frac{1}{1 - e^{-cs}} \int_0^c e^{-st} f(t)\, dt\ .$$

b) Do Exercise 3C-3, using the above formula.

3D-4. Find $\mathcal{L}^{-1}$ by using the convolution: a) $\dfrac{s}{(s+1)(s^2+4)}$ b) $\dfrac{1}{(s^2+1)^2}$

Your answer should not contain the convolution $*$.

3D-5. Assume $f(t) = 0$, for $t \le 0$. Show informally that $\delta(t) * f(t) = f(t)$, by using the definition of convolution; then do it by using the definition of $\delta(t)$.
(See (5), section 4.6 of your book; $\delta(t)$ is written $\delta_0(t)$ there.)

3D-6. Prove that $f(t) * g(t) = g(t) * f(t)$ directly from the definition of convolution, by making a change of variable in the convolution integral.

3D-7. Show that the IVP: $y'' + k^2 y = r(t), \quad y(0) = 0,\ y'(0) = 0$ has the solution

$$y(t) = \frac{1}{k} \int_0^t r(u) \sin k(t-u)\, du\ ,$$

by using the Laplace transform and the convolution.

3D-8. By using the Laplace transform and the convolution, show that in general the IVP (here a and b are constants):

$$y'' + ay' + by = r(t), \qquad y(0) = 0,\ y'(0) = 0,$$

has the solution

$$y(t) = \int_0^t w(t-u) r(u)\, du\ ,$$

where $w(t)$ is the solution to the IVP: $y'' + ay' + by = 0, \quad y(0) = 0,\ y'(0) = 1$.

(The function $w(t-u)$ is called the **Green's function** for the linear operator $D^2 + aD + b$.)