Section 3 Solutions

$\boxed{3A-1}$ $\mathcal{L}\{t\} = \int_0^\infty t e^{-st}\,dt$. Integrate by parts:

$$= t\frac{e^{-st}}{-s}\Big]_{t=0}^{\infty} - \int_0^\infty \frac{e^{-st}}{-s}\,dt$$

Since $\lim_{t\to\infty} te^{-st} = 0$ $\boxed{\text{if } s>0,}$ the left-hand term is 0 at both endpoints. Integrating the right-hand term:

$$= -\frac{e^{-st}}{(-s)(-s)}\Big]_0^\infty = 0 - \left(\frac{-1}{s^2}\right) = \frac{1}{s^2}, \quad s>0.$$

$\boxed{3A-2}$ $\mathcal{L}\{e^{(a+ib)t}\} = \mathcal{L}\{e^{at}\cos bt\} + i\mathcal{L}\{e^{at}\sin bt\}$ ⊛

On the other hand,

$\mathcal{L}\{e^{(a+ib)t}\} = \dfrac{1}{s-(a+ib)}$; multiplying top + bottom by $(s-a)+ib$:

$$= \frac{(s-a)+ib}{(s-a)^2+b^2} = \frac{s-a}{(\cdots)} + \frac{ib}{(\cdots)} \quad ⊛⊛$$

$$\therefore \mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}, \quad \mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$$

[by equating real + imag. parts of ⊛ and ⊛⊛.]

$\boxed{3A-3}$ a) $\mathcal{L}^{-1}\left(\dfrac{1}{\frac{s}{2}+3}\right) = \mathcal{L}^{-1}\left(\dfrac{2}{s+6}\right) = 2e^{-6t}$

b) $\mathcal{L}^{-1}\left(\dfrac{3}{s^2+4}\right) = \dfrac{3}{2}\mathcal{L}^{-1}\left(\dfrac{2}{s^2+4}\right) = \dfrac{3}{2}\sin 2t$

c) $\mathcal{L}^{-1}$: $\dfrac{1}{s^2-4} = \dfrac{1/4}{s-2} - \dfrac{1/4}{s+2}$ (partial fractions)

$$\therefore \mathcal{L}^{-1}\left(\frac{1}{s^2-4}\right) = \frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t}$$

d) $\dfrac{1+2s}{s^3} = \dfrac{1}{s^3} + \dfrac{2}{s^2}$

$$\therefore \mathcal{L}^{-1}\left(\frac{1+2s}{s^3}\right) = \frac{t^2}{2} + 2t$$

e) $\dfrac{1}{s^4-9s^2} = \dfrac{-1/9}{s^2} + \dfrac{0}{s} + \dfrac{1/54}{s-3} + \dfrac{-1/54}{s+3}$

$= \dfrac{1}{s^2(s-3)(s+3)}$ (by cover-up method. Find the 0 by putting $s=1$)

$$\therefore \mathcal{L}^{-1}\left(\frac{1}{s^4-9s^2}\right) = -\frac{t}{9} + \frac{1}{54}\left(e^{3t} - e^{-3t}\right)$$

$\boxed{3A-4}$ $\mathcal{L}\{\sin at\} = \int_0^\infty \sin at \cdot e^{-st}\,dt$; Integrate by parts:

$$= \sin at \cdot \frac{e^{-st}}{-s}\Big]_0^\infty - \int_0^\infty a\cos at \cdot \frac{e^{-st}}{-s}\,dt$$

$$= 0 + \frac{a}{s}\mathcal{L}\{\cos at\}$$

$$= \frac{a}{s}\cdot\frac{s}{s^2+a^2}, \quad s>0$$

$$= \frac{a}{s^2+a^2}, \quad s>0.$$

$\boxed{3A-5}$ $\mathcal{L}\{\cos^2 at\} = \mathcal{L}\{\frac{1}{2} + \frac{1}{2}\cos 2at\}$

$$= \mathcal{L}\{\tfrac{1}{2}\} + \tfrac{1}{2}\mathcal{L}\{\cos 2at\}$$

$$= \frac{1}{2s} + \frac{1}{2}\left(\frac{s}{s^2+4a^2}\right).$$

$$\mathcal{L}\{\sin^2 at\} = \mathcal{L}\{\tfrac{1}{2} - \tfrac{1}{2}\cos 2at\}$$

$$= \frac{1}{2s} - \frac{1}{2}\left(\frac{s}{s^2+4a^2}\right)$$

$\mathcal{L}\{\cos^2 at + \sin^2 at\} = \dfrac{1}{s}$, from the above;

$$\mathcal{L}\{1\} = \frac{1}{s} \ ✓.$$

$\boxed{3A-6}$ a) $\mathcal{L}\left\{\dfrac{1}{\sqrt{t}}\right\} = \int_0^\infty e^{-st}\dfrac{1}{\sqrt{t}}\,dt$, $(s>0)$

Put $x^2 = st$, so $t = \dfrac{x^2}{s}$

Then the integral becomes (in terms of s, x):

$$= \int_0^\infty e^{-x^2}\frac{\sqrt{s}}{x}\cdot\frac{2x\,dx}{s}$$

$$= \frac{2}{\sqrt{s}}\int_0^\infty e^{-x^2}dx = \frac{2}{\sqrt{s}}\cdot\frac{\sqrt{\pi}}{2}$$

$$= \sqrt{\pi/s}.$$

b) $\mathcal{L}\{\sqrt{t}\} = \int_0^\infty e^{-st}\sqrt{t}\,dt$; integrate by parts:

$$= \sqrt{t}\,\frac{e^{-st}}{-s}\Big]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\cdot\frac{1}{2\sqrt{t}}\,dt$$

$$= 0 + \frac{1}{2s}\int_0^\infty e^{-st}\cdot\frac{1}{\sqrt{t}}\,dt$$

$$= \frac{1}{2s}\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \frac{1}{2s}\cdot\sqrt{\frac{\pi}{s}} = \frac{\sqrt{\pi}}{2s^{3/2}}.$$

3A-7 $\mathcal{L}\{e^{t^2}\} = \int_0^\infty e^{-st}\cdot e^{t^2}\,dt$
$$= \int_0^\infty e^{t^2-st}\,dt$$

This integral is infinite for every real value of s, no matter how large, since if $t > s$, $t^2 - st > 0$, and therefore
$$\int_0^\infty e^{t^2-st}\,dt > \int_s^\infty e^{t^2-st}\,dt > \int_s^\infty e^0\,dt = \infty.$$

3A-8 $\mathcal{L}\{\frac{1}{t^k}\} = \int_0^\infty e^{-st}\frac{1}{t^k}\,dt$, $(s>0)$

The trouble here is when $t=0$.
Near $t=0$, $e^{-st} \approx e^0 = 1$.
$\therefore$ the integral is like:
$$\int_0^a e^{-st}\frac{1}{t^k}\,dt \gtrsim \int_0^a \frac{dt}{t^k}$$
and this last integral converges only if $k<1$ $\left[\text{since it } \begin{cases} = \left.\frac{t^{1-k}}{1-k}\right|_0^a & \text{for } k \neq 1 \\ = \left.\ln x\right|_0^a & \text{for } k=1 \end{cases}\right]$

[At the upper limit ∞ the original integral always converges, if $s>0$].
$\therefore$ $\mathcal{L}\{\frac{1}{t^k}\}$ exists for $k<1$.

3A-9 a) $\mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9} = F(s)$
By the exponential-shift formula,
$$\mathcal{L}\{e^{-t}\sin 3t\} = F(s+1) = \frac{3}{(s+1)^2+9}$$

b) $\mathcal{L}\{t^2-3t+2\} = \frac{2}{s^3} - \frac{3}{s^2} + \frac{2}{s} = F(s)$
By exponential-shift rule,
$$\mathcal{L}\{e^{2t}(t^2-3t+2)\} = F(s-2)$$
$$= \frac{2}{(s-2)^3} - \frac{3}{(s-2)^2} + \frac{2}{s-2}$$

3A-10 $\mathcal{L}^{-1}\{\frac{3}{(s-2)^4}\} = e^{2t}\mathcal{L}^{-1}\{\frac{3}{s^4}\} =$
$$= e^{2t}\frac{t^3}{2}$$

$$\mathcal{L}^{-1}\{\frac{1}{s(s-2)}\} = \mathcal{L}^{-1}\{\frac{1/2}{s-2} - \frac{1/2}{s}\},$$
(by partial fractions)
$$= \tfrac{1}{2}e^{2t} - \tfrac{1}{2}.$$

$\mathcal{L}^{-1}\{\frac{s+1}{s^2-4s+5}\}$:

Complete the square in the denominator:
$$\frac{s+1}{s^2-4s+5} = \frac{s+1}{(s-2)^2+1};$$ express top in terms of $s-2$:
$$= \frac{s-2}{(s-2)^2+1} + \frac{3}{(s-2)^2+1}$$
$$\therefore \mathcal{L}^{-1}(\cdots) = e^{2t}\cos t + 3e^{2t}\sin t,$$
(by the exponential-shift rule).

3B-1 We use throughout the two formulas:

$$\mathcal{L}(y') = -y(0+) + sY \leftarrow (\mathcal{L}(y))$$

and

$$\mathcal{L}(y'') = -y'(0+) - s\,y(0+) + s^2 Y$$

[The 0+ indicates that if $y(t)$ is discontinuous at 0, we $\lim_{t\to 0+} y(t)$, the righthand limit].

a) $y' - y = e^{3t}, \quad y(0) = 1$

$$(sY - 1) - Y = \frac{1}{s-3}$$
$$(s-1)Y = 1 + \frac{1}{s-3}$$
$$Y = \frac{1}{s-1} + \frac{1}{(s-3)(s-1)}$$

make partial fractions decomp; $= \dfrac{1/2}{s-1} + \dfrac{1/2}{s-3}$

$$\therefore \boxed{y = \tfrac{1}{2}e^{t} + \tfrac{1}{2}e^{3t}}$$

b) $y'' - 3y' + 2y = 0, \quad y(0)=1, y'(0)=1$

$$(s^2Y - s - 1) - 3(sY - 1) + 2Y = 0$$
$$\therefore (s^2 - 3s + 2)Y = s - 2$$
$$Y = \frac{1}{s-1}$$
$$\therefore \boxed{y = e^{t}}$$

c) $y'' + 4y = \sin t, \quad y(0)=1, y'(0)=0$

$$(s^2Y - s) + 4Y = \frac{1}{s^2+1}$$
$$\therefore Y = \frac{1}{(s^2+1)(s^2+4)} + \frac{s}{s^2+4} \quad (*)$$

Apply partial fractions ↑; treat s^2 as a single variable: i.e.,

$$\frac{1}{(u+1)(u+4)} = \frac{1/3}{u+1} - \frac{1/3}{u+4}\,; \quad \text{now put } u = s^2.$$

$$Y = \frac{1/3}{s^2+1} - \frac{1/3}{s^2+4} + \frac{s}{s^2+4}$$

$$\therefore \boxed{y = \tfrac{1}{3}\sin t - \tfrac{1}{6}\sin 2t + \cos 2t}$$

(*) Note that it's easier not to combine terms at this point

d) $y'' - 2y' + 2y = 2e^{t}, \quad y(0)=0, \; y'(0)=1$

$$(s^2Y - 1) - 2sY + 2Y = \frac{2}{s-1}$$
$$\therefore (s^2 - 2s + 2)Y = \frac{2}{s-1} + 1 = \frac{s+1}{s-1}$$
$$Y = \frac{s+1}{(s^2-2s+2)(s-1)}$$

By partial fractions:

$$Y = \frac{2}{s-1} + \frac{3-2s}{s^2-2s+2}\,; \quad \text{complete the square:}$$
$$= \frac{2}{s-1} - \frac{2(s-1)}{(s-1)^2+1} + \frac{1}{(s-1)^2+1}$$

(note how we write the 2^{nd} term as an expression in $s-1$; the last term is what's left over.)

$$\therefore y = 2e^{t} - 2e^{t}\cos t + e^{t}\sin t$$

e) $y'' - 2y' + y = e^{t}, \quad y(0)=1, \; y'(0)=0.$

$$s^2Y - s - 2(sY - 1) + Y = \frac{1}{s-1}$$
$$\underbrace{(s^2 - 2s + 1)}_{(s-1)^2}Y = \frac{1}{s-1} + s - 2$$
$$= \frac{1}{s-1} + (s-1) - 1$$
$$\therefore Y = \frac{1}{(s-1)^3} + \frac{1}{(s-1)} - \frac{1}{(s-1)^2}$$
$$\therefore \boxed{y = \tfrac{t^2}{2}e^{t} + e^{t} - te^{t}}$$

3B-2

12. $\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$

Integ. by parts: $= e^{-st}f(t)\Big]_0^\infty - \int_0^\infty -se^{-st}f(t)\,dt$

see below ↓ (since $f(t)$ is of exp. order) $\Rightarrow 0 - f(0) + s\int_0^\infty e^{-st}f(t)\,dt$

$$\therefore \mathcal{L}\{f'(t)\} = -f(0) + s\mathcal{L}\{f(t)\}.$$

Assumes:

$f'(t)$ piecewise continuous and exponential order (so $\int_0^\infty e^{-st}f'(t)\,dt$ exists) (i.e., $|f'(t)| < Ke^{at}$ if t is large).

$f(t)$ of exponential order, so $\mathcal{L}\{f\}$ exists. (It's continuous, since $f'(t)$ exists.)

3B-3

These use the formula:

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} \underset{=\mathcal{L}\{f(t)\}}{F(s)}$$

a)

$$\mathcal{L}\{t \cos bt\} = (-1)\frac{d}{ds}\left(\frac{s}{s^2+b^2}\right)$$
$$= \frac{b^2 - s^2}{(b^2+s^2)^2}$$

b) $\mathcal{L}\{t^n e^{kt}\}$: by the exp-shift rule,

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$
$$\therefore \mathcal{L}\{t^n e^{kt}\} = \frac{n!}{(s-k)^{n+1}}.$$

By the above formula,

$$\mathcal{L}\{t^n e^{kt}\} = (-1)^n \frac{d^n}{ds^n}(s-k)^{-1}$$
$$= (-1)^n \cdot (-1)(-2)\cdots(-n)(s-k)^{-(n+1)}$$
$$= \frac{n!}{(s-k)^{n+1}}, \text{ as before.}$$

c) $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$

$$\mathcal{L}\{t \sin t\} = \frac{2s}{(s^2+1)^2} \quad \text{by the above fmla.}$$

$$\therefore \mathcal{L}\{te^{at}\sin t\} = \frac{2(s-a)}{((s-a)^2+1)^2}.$$

3B-4

a) $\mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^2}\right) = \frac{t \sin t}{2}$,

as in 13 (c) above

b) $\frac{1}{(s^2+1)^2}$ suggests some combination of $\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$ and $\frac{d}{ds}\left(\frac{s}{s^2+1}\right)$

$$\mathcal{L}\{t\cos t\} = -\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = \frac{1}{s^2+1} - \underset{\text{what we want}}{\frac{2}{(s^2+1)^2}}$$
$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1} \nearrow$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \frac{1}{2}[\sin t - t\cos t]$$

3B-5

a) $\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t)\,dt$

$$= \int_0^\infty e^{-(s-a)t}f(t)\,dt$$
$$= F(s-a),$$

since $F(s) = \int_0^\infty e^{-st}f(t)\,dt$.

b) $F(s) = \int_0^\infty e^{-st}f(t)\,dt$

Differentiating under the integral sign, with respect to s: (*)

$$F'(s) = \int_0^\infty -te^{-st}f(t)\,dt,$$

since t is a constant with respect to the differentiation;

$$= \mathcal{L}\{-tf(t)\}$$
$$= -\mathcal{L}\{tf(t)\}.$$

[(*) this is legal if $f(t)$ is continuous and of exponential order].

3B-6

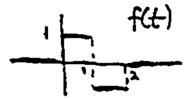
$$y'' + ty = 0, \quad y(0)=1, \; y'(0)=0$$

Take the Laplace transform:

$$(s^2 Y - s) - \frac{d}{ds}Y = 0$$
$$\frac{dY}{ds} - s^2 Y = -s,$$

(which is first order, linear).

$\boxed{3C-1}$

a) 

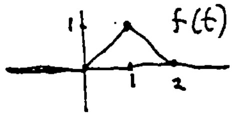
Using $u(t)$: $f(t) = u(t) - 2u(t-1) + u(t-2)$

$\therefore F(s) = \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s} = \frac{1}{s}(1-e^{-s})^2$

Directly:

$F(s) = \int_0^1 e^{-st}dt - \int_1^2 e^{-st}dt = \frac{1}{s}(1-e^{-s})^2$

(by straight calc.)

b) 

Using $u(t)$: $f(t) = t\, u(t) - u(t-1)\cdot 2(t-1)$
$\quad + u(t-2)(t-2)$

$\therefore F(s) = \frac{1}{s^2}(1-2e^{-s}+e^{-2s})$ $\quad$ (*)

Directly:

$F(s) = \int_0^1 te^{-st}dt + \int_1^2 (2-t)e^{-st}dt$ $\quad$ [integrate each $\int$ by parts]

$= \frac{te^{-st}}{-s}\Big]_0^1 - \left[\frac{e^{-st}}{(-s)^2}\right]_0^1 + (2-t)\frac{e^{-st}}{-s}\Big]_1^2 - \frac{-e^{-st}}{(-s)^2}\Big]_1^2$ $\quad$ (which agrees with * after canceling terms)

c) 

$|\sin t| = (-1)^n \sin t,$

$n\pi \le t \le (n+1)\pi.$

This can be done directly,
(adding up the integrals over even + odd intervals):

$$F(s) = \int_0^\infty |\sin t| e^{-st}dt = \sum_{n=0}^\infty \int_{n\pi}^{(n+1)\pi} (-1)^n \sin t \cdot e^{-st}dt$$

Change variable: $u = t - n\pi$

$$= \sum_{n=0}^\infty \int_0^\pi (-1)^n \sin(u+n\pi)e^{-s(u+n\pi)}du$$

$\sin(u+n\pi) = (-1)^n \sin u$; $e^{-sn\pi}$ is a "constant"

$$= \sum_{n=0}^\infty e^{-sn\pi} \underbrace{\int_0^\pi \sin u \cdot e^{-su}du}_{\text{call it "K".}}$$

Then $K = \frac{1+e^{-s\pi}}{1+s^2}$ (from tables)

$= K \cdot \sum_0^\infty e^{-sn\pi}$; adding up this geometric series gives

$= K \cdot \frac{1}{1-e^{-s\pi}}$ $\qquad$ <u>Ans:</u> $\frac{1+e^{-s\pi}}{(1+s^2)(1-e^{-s\pi})}$

$\boxed{3C-2}$

a) $\frac{1}{s^2+3s+2} = \frac{1}{s+1} - \frac{1}{s+2}$ (partial fractions)

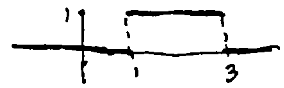
$\mathcal{L}^{-1}\left\{\frac{1}{s^2+3s+2}\right\} = e^{-t} - e^{-2t} = f(t)$

$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2+3s+2}\right\} = u(t-1)f(t-1)$

$= u(t-1)\left(e^{1-t} - e^{2(1-t)}\right)$

b) $\mathcal{L}^{-1}\left(\frac{e^{-s}-e^{-3s}}{s}\right) = \mathcal{L}^{-1}\left(\frac{e^{-s}}{s}\right) - \mathcal{L}^{-1}\left(\frac{e^{-3s}}{s}\right)$

$= u(t-1) - u(t-3)$



$\boxed{3C-3}$

a) $\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$

$= \int_0^1 e^{-st}dt + \int_2^3 e^{-st}dt + \int_4^5 e^{-st}dt + \ldots$

$= \frac{e^0 - e^{-s}}{s} + \frac{e^{-2s}-e^{-3s}}{s} + \frac{e^{-4s}-e^{-5s}}{s} + \ldots$

$= \frac{1}{s} \cdot (e^0 - e^{-s} + e^{-2s} - e^{-3s} + \ldots)$

geometric series, whose sum is

$= \frac{1}{s} \cdot \left(\frac{1}{1+e^{-s}}\right)$

b) $f(t) = u(t) - u(t-1) + u(t-2) - \ldots$

$\therefore \mathcal{L}\{f(t)\} = \frac{1}{s} - \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \ldots$

$= \frac{1}{s}(e^0 - e^{-s} + e^{-2s} - e^{-3s} \ldots)$

$= \frac{1}{s} \cdot \frac{1}{1+e^{-s}}$, as before.

3C-4

$$\mathcal{L}\{h(t)\} = \mathcal{L}\{u(t-\pi) - u(t-2\pi)\}$$
$$= \frac{e^{-s\pi} - e^{-2s\pi}}{s}$$

The ODE is: $y'' + 2y' + 2y = h(t)$, $\quad y(0) = 0,\ y'(0) = 1$;

Laplace Transform is:

$$(s^2Y - 1) + 2(sY) + 2Y = \frac{e^{-s\pi} - e^{-2s\pi}}{s}$$
$$(s^2 + 2s + 2)Y = 1 + \frac{e^{-s\pi} - e^{-2s\pi}}{s}$$
$$Y = \frac{1}{(s+1)^2 + 1}\left[1 + \frac{e^{-s\pi} - e^{-2s\pi}}{s}\right]$$

By partial fractions

$$\frac{1}{(s^2+2s+2)s} = \frac{-s/2 - 1}{s^2+2s+2} + \frac{1/2}{s}$$
$$= \frac{-\frac{1}{2}(s+1) - \frac{1}{2}}{(s+1)^2 + 1} + \frac{1/2}{s}$$

$$\therefore\ y = e^{-t}\sin t$$
$$+ \frac{1}{2}\Big[1 - e^{(t-\pi)}\big(\underbrace{\sin(t-\pi)}_{= -\sin t} + \underbrace{\cos(t-\pi)}_{= -\cos t}\big)\Big]u(t-\pi)$$
$$- \frac{1}{2}\Big[1 - e^{(t-2\pi)}\big(\underbrace{\sin(t-2\pi)}_{= \sin t} + \underbrace{\cos(t-2\pi)}_{= \cos t}\big)\Big]u(t-2\pi)$$

$$\therefore\ y = \begin{cases} e^{-t}\sin t, & (0 \le t \le \pi) \\ \frac{1}{2} + \left(1 + \frac{e^{\pi}}{2}\right)e^{-t}\sin t + \frac{e^{\pi}}{2}e^{-t}\cos t, & (\pi \le t \le 2\pi) \\ \left(1 + \frac{e^{\pi}}{2} + \frac{e^{2\pi}}{2}\right)e^{-t}\sin t + \left(\frac{e^{\pi}}{2} + \frac{e^{2\pi}}{2}\right)e^{-t}\cos t, & (2\pi \le t) \end{cases}$$

3C-5

$$\mathcal{L}\{\overbrace{u(t)\cdot t}^{=r(t)}\} = \mathcal{L}\{t\} = \frac{1}{s^2}$$

$y'' - 3y' + 2y = r(t)$, $\quad y(0) = 1,\ y'(0) = 0$ gives:

$$(s^2Y - s) - 3(sY - 1) + 2Y = \frac{1}{s^2}$$
$$(s^2 - 3s + 2)Y = s - 3 + \frac{1}{s^2}$$
$$Y = \frac{s-3}{(s-2)(s-1)} + \frac{1}{s^2(s-2)(s-1)}$$
$$= \frac{s^3 - 3s^2 + 1}{s^2(s-2)(s-1)}$$

cont'd above

3C-5 (cont'd) By partial fractions

$$Y = \frac{1}{s-1} - \frac{3/4}{s-2} + \frac{3/4}{s} + \frac{1/2}{s^2}$$
$$\therefore\ y = e^{t} - \frac{3}{4}e^{2t} + 3/4 + t/2$$

3D-1

$$y'' + 2y' + y = \delta(t) + u(t-1) \qquad y(0) = 0,\ y'(0) = 1$$
$$(s^2Y - 1) + 2sY + Y = 1 + \frac{e^{-s}}{s}$$
$$(s^2 + 2s + 1)Y = 2 + \frac{e^{-s}}{s};$$

Divide, use part. fractions:

$$Y = \frac{2}{(s+1)^2} + e^{-s}\left[\frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}\right]$$
$$y = 2te^{-t} + u(t-1)\left[1 - e^{-(t-1)} - (t-1)e^{-(t-1)}\right]$$
$$= 2te^{-t} + u(t-1)\left[1 - te^{1-t}\right]$$
$$\therefore\ y(t) = \begin{cases} 2te^{-t}, & 0 \le t \le 1 \\ 1 + (2-e)te^{-t}, & t \ge 1 \end{cases}$$

3D-2

$$y'' + y = r(t), \qquad y(0) = 0,\ y'(0) = 1$$
$$r(t) = \begin{cases} 1, & 0 \le x \le \pi \\ 0 & \text{otherwise} \end{cases}$$
$$= 1 - u(t-\pi)$$
$$\therefore\ \mathcal{L}\{r(t)\} = \frac{1}{s} - \frac{e^{-\pi s}}{s}$$

So

$$(s^2Y - 1) + Y = \frac{1 - e^{-\pi s}}{s}$$
$$Y = \frac{1}{s^2+1} + \frac{1}{s(s^2+1)} - \frac{e^{-\pi s}}{s(s^2+1)}$$

$$\boxed{\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}}$$

$$y = \sin t + 1 - \cos t - \big(1 - \underbrace{\cos(t-\pi)}_{[= -\cos t]}\big)u(t-\pi)$$
$$\therefore\ y = \begin{cases} 1 + \sin t - \cos t, & 0 \le t \le \pi \\ \sin t - 2\cos t, & t \ge \pi \end{cases}$$

3D-3

Also: $\boxed{\text{SEE BELOW}}$

a) $F(s) = \int_0^\infty e^{-st} f(t)\,dt$

$= \sum_{n=0}^{\infty} \int_{nc}^{(n+1)c} e^{-st} f(t)\,dt$

[breaking $[0,\infty)$ up into the intervals $[nc, (n+1)c]$.

Change variable: $u = t - nc$

$$\int_{nc}^{(n+1)c} e^{-st} f(t)\,dt = \int_0^c e^{-s(u+nc)} f(u)\,du,$$

since $f(u+nc) = f(u)$.

Therefore our sum becomes:

$$F(s) = \sum_{n=0}^{\infty} e^{-snc} \underbrace{\int_0^c e^{-su} f(u)\,du}_{\text{call this } K}$$

$$= K \sum_{n=0}^{\infty} (e^{-sc})^n \leftarrow \text{a geometric series, whose sum is}$$

$$= K \cdot \frac{1}{1-e^{-sc}}$$

$$\therefore\ F(s) = \frac{1}{1-e^{-sc}} \cdot \int_0^c e^{-su} f(u)\,du$$

(FOR A BETTER WAY, SEE NEXT PAGE)

b) For problem 19, $c = 2$

$$\int_0^2 e^{-su} f(u)\,du = \int_0^1 e^{-su}\,du$$

$$= \frac{1-e^{-s}}{s}$$

$$\therefore\ F(s) = \frac{1}{1-e^{-2s}} \cdot \frac{1-e^{-s}}{s}$$

$$= \frac{1}{s\cdot(1+e^{-s})}, \text{ as before.}$$

3D-4

a) $\dfrac{s}{(s+1)(s^2+4)} = \dfrac{1}{s+1} \cdot \dfrac{s}{s^2+4}$

$$\therefore\ \mathcal{L}^{-1}\left(\cdots\right) = e^{-t} * \cos 2t$$

$$= \int_0^t e^{-(t-u)} \cos 2u\,du$$

$$= e^{-t} \int_0^t e^{u} \cos 2u\,du$$

$$= e^{-t}\left[\frac{e^t}{5}(\cos 2t + 2\sin 2t) - \frac{1}{5}\right]$$

$$= \frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t - \frac{1}{5}e^{-t}$$

b) $\dfrac{1}{(s^2+1)^2} = \dfrac{1}{s^2+1} \cdot \dfrac{1}{s^2+1}$

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right) = \sin t * \sin t$$

$$= \int_0^t \sin(t-u)\cdot \sin u\,du$$

Easiest is to use a trig identity:

$$= \int_0^t \frac{1}{2}\left[\cos(t-2u) - \cos t\right] du$$

$$= \frac{\sin t}{2} - \frac{t}{2}\cos t.$$

3D-5

a) $f(t) \overset{\mathcal{L}}{\rightsquigarrow} F(s)$, $\delta(t) \overset{\mathcal{L}}{\rightsquigarrow} 1$

$$\mathcal{L}\{\delta * f\} = 1\cdot F(s) = F(s)$$

$$\therefore\ \delta * f(t) = f(t)u(t) = f(t),$$

[THIS IS JUST FORMAL] since $f(t) = 0$, $t \le 0$.

b) Using the definition of $*$:

$$\delta * f = \int_0^t \delta(t-u) f(u)\,du$$

$$= \int_{-\infty}^{\infty} \delta(t-u) f(u)\,du$$

(SHADY) $= f(t)\displaystyle\int_{-\infty}^{\infty} \delta(t-u)\,du$ — since $\delta(t-u) = 0$ except if $u = t$

$$= f(t)\cdot 1$$

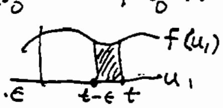
✓

c) using the "definition" of $\delta(t)$

$$\delta * f(t) = \int_0^t \delta(t-u) f(u)\,du = \int_0^t \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[u(t-u_1) - u(t-u_1-\epsilon)\right] f(t)\,dt$$

$$= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_0^t \left[u(t-u_1) - u(t-u_1-\epsilon)\right] f(t)\,dt = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int_0^t f(u_1)\,du_1 - \int_0^{t-\epsilon} f(u_1)\,du_1\right]$$

(SHADY!) $= \displaystyle\lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_{t-\epsilon}^{t} f(u_1)\,du = f(t)$, since area $\approx f(t)\cdot\epsilon$



$\boxed{3D-6}$

$$(f * g)(t) = \int_0^t f(t-u)\,g(u)\,du$$

let $x = t-u$ (change variable u to the var. x in the integral)
$dx = -du$

limits:
when $u=0$, $x=t$
when $u=t$, $x=0$
$\therefore$ integral becomes:

$$= -\int_t^0 f(x)\,g(t-x)\,dx = \int_0^t g(t-x)\,f(x)\,dx$$

$$= (g*f)(t).$$

$\boxed{3D-7}$

Taking the Laplace transform:

$$s^2Y + k^2Y = R(s),$$

where $R(s) = \mathcal{L}\{r(t)\}$.

$$\therefore\ Y = \frac{R(s)}{s^2+k^2} = \frac{1}{s^2+k^2}\cdot R(s)$$

$$\therefore\ y = \frac{1}{k}\sin kt * r(t)$$

$$= \frac{1}{k}\int_0^t \sin k(t-u)\cdot r(u)\,du.$$

$\boxed{3D-8}$

$$y'' + ay' + by = r(t), \quad y(0)=0,\ y'(0)=0$$

$$s^2Y + asY + bY = R(s)$$

$$\therefore\ Y = \frac{1}{s^2+as+b}\cdot R(s)$$

$$\therefore\ y = g(t) * r(t), \quad \text{where} \quad g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+as+b}\right\}$$

$$y = \int_0^t g(t-u)\,r(u)\,du.$$

To interpret $g(t)$, consider the ODE-IVP

$$y'' + ay' + by = 0, \quad y(0)=0,\ y'(0)=1.$$

then $s^2Y - 1 + asY + bY = 0$

so $Y = \dfrac{1}{s^2+as+b}$

and $y = g(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{s^2+as+b}\right\}$. Thus $g(t)$ may be interpreted as the soln to this IVP.

(see above for another interp. of g)

$\boxed{3D-8}$ (continued)

$g(t)$ may also be interpreted as the solution to

$$y'' + ay' + by = \delta(t), \quad y(0)=0,\ y'(0)=0$$

since this leads to

$$s^2Y + asY + bY = 1$$

or $Y = \dfrac{1}{s^2+as+b}$,

so that $y = g(t)$.

$\boxed{3D-3}$

a.

we have:

$$u(t-c)f(t-c) + f_0(t) = u(t)f(t),$$

where $f_0(t) = \begin{cases} f(t), & 0 \le t < c \\ 0 & \text{elsewhere} \end{cases}$.

$\therefore$ taking LT's:

$$e^{-cs}F(s) + \int_0^c e^{-st}f(t)\,dt = F(s).$$

Solve for $F(s)$:

$$F(s) = \frac{1}{1-e^{-cs}}\int_0^c e^{-st}f(t)\,dt.$$