6. Power Series

6A. Power Series Operations

6A-1. Find the radius of convergence for each of the following:

a) $\displaystyle\sum_0^\infty n\,x^n$ b) $\displaystyle\sum_0^\infty \frac{x^{2n}}{n2^n}$ c) $\displaystyle\sum_1^\infty n!\,x^n$ d) $\displaystyle\sum_0^\infty \frac{(2n)!}{(n!)^2}\,x^n$

6A-2. Starting from the series $\displaystyle\sum_0^\infty x^n = \frac{1}{1-x}$ and $\displaystyle\sum_0^\infty \frac{x^n}{n!} = e^x$,

by using operations on series (substitution, addition and multiplication, term-by-term differentiation and integration), find series for each of the following

a) $\dfrac{1}{(1-x)^2}$ b) xe^{-x^2} c) $\tan^{-1} x$ d) $\ln(1+x)$

6A-3. Let $\displaystyle y = \sum_0^\infty \frac{x^{2n+1}}{(2n+1)!}$. Show that

a) y is a solution to the ODE $y'' - y = 0$ b) $y = \sinh x = \frac{1}{2}(e^x - e^{-x})$.

6A-4. Find the sum of the following power series (using the operations in 6A-2 as a help):

a) $\displaystyle\sum_0^\infty x^{3n+2}$ b) $\displaystyle\sum_0^\infty \frac{x^n}{n+1}$ c) $\displaystyle\sum_0^\infty nx^n$

6B. First-order ODE's

6B-1. For the nonlinear IVP $y' = x + y^2$, $y(0) = 1$, find the first four nonzero terms of a series solution $y(x)$ two ways:

a) by setting $y = \sum_0^\infty a_n x^n$ and finding in order $a_0, a_1, a_2, \dots$, using the initial condition and substituting the series into the ODE;

b) by differentiating the ODE repeatedly to obtain $y(0), y'(0), y''(0), \dots$, and then using Taylor's formula.

6B-2. Solve the following linear IVP by assuming a series solution

$$y = \sum_0^\infty a_n x^n ,$$

substituting it into the ODE and determining the a_n recursively by the method of undetermined coefficients. Then sum the series to obtain an answer in closed form, if possible (the techniques of 6A-2,4 will help):

a) $y' = x + y$, $y(0) = 0$ b) $y' = -xy$, $y(0) = 1$ c) $(1-x)y' - y = 0$, $y(0) = 1$

6C. Solving Second-order ODE's

6C-1. Express each of the following as a power series of the form $\sum_N^\infty b_n x^n$. Indicate the value of N, and express b_n in terms of a_n.

a) $\sum_1^\infty a_n x^{n+3}$ b) $\sum_0^\infty n(n-1)a_n x^{n-2}$ c) $\sum_1^\infty (n+1)a_n x^{n-1}$

6C-2. Find two independent power series solutions $\sum a_n x^n$ to $y'' - 4y = 0$, by obtaining a recursion formula for the a_n.

6C-3. For the ODE $y'' + 2xy' + 2y = 0$,

a) find two independent series solutions y_1 and y_2;

b) determine their radius of convergence;

c) express the solution satisfying $y(0) = 1$, $y'(0) = -1$ in terms of y_1 and y_2;

d) express the series in terms of elementary functions (i.e., sum the series to an elementary function).

(One of the two series is easily recognizable; the other can be gotten using the operations on series, or by using the known solution and the method of reduction of order—see Exercises 2B.)

6C-4. Hermite's equation is $y'' - 2xy' + ky = 0$. Show that if k is a positive even integer $2m$, then one of the power series solutions is a polynomial of degree m.

6C-5. Find two independent series solutions in powers of x to the Airy equation: $y'' = xy$.

Determine their radius of convergence. For each solution, give the first three non-zero terms and the general term.

6C-6. Find two independent power series solutions $\sum a_n x^n$ to

$$(1 - x^2)y'' - 2xy' + 6y = 0 \ .$$

Determine their radius of convergence R. To what extent is R predictable from the original ODE?

6C-7. If the recurrence relation for the a_n has three terms instead of just two, it is more difficult to find a formula for the general term of the corresponding series. Give the recurrence relation and the first three nonzero terms of two independent power series solutions to

$$y'' + 2y' + (x - 1)y = 0 \ .$$