10. Operators and the Exponential Response Formula

10.1. **Operators.** Operators are to functions as functions are to numbers. An operator takes a function, does something to it, and returns this modified function. There are lots of examples of operators around:

—The *shift-by-a operator* (where a is a number) takes as input a function $f(t)$ and gives as output the function $f(t-a)$. This operator shifts graphs to the right by a units.

—The *multiply-by-$h(t)$ operator* (where $h(t)$ is a function) multiplies by $h(t)$: it takes as input the function $f(t)$ and gives as output the function $h(t)f(t)$.

You can go on to invent many other operators. In this course the most important operator is:

—The *differentiation operator*, which carries a function $f(t)$ to its derivative $f'(t)$.

The differentiation operator is usually denoted by the letter D; so $Df(t)$ is the function $f'(t)$. D has carried f to f'; for example, $Dt^3 = 3t^2$. Warning: you can't take this equation and substitute $t = 2$ to get $D8 = 12$. The only way to interpret 8 is as a *constant* function, which of course has derivative zero: $D8 = 0$. The point is that in order to know the function $Df(t)$ at a particular value of t, say $t = a$, you need to know more than just $f(a)$; you need to know how $f(t)$ is changing near a as well. This is characteristic of operators; in general you have to expect to need to know the *whole* function $f(t)$ in order to evaluate an operator on it.

The *identity operator* takes an input function $f(t)$ and returns the *same* function, $f(t)$; it does nothing, but it still gets a symbol, I.

Operators can be added and multiplied by numbers or more generally by functions. Thus $tD+4I$ is the operator sending $f(t)$ to $tf'(t)+4f(t)$.

The single most important thing associated with the concept of operators is that they can be *composed* with each other. I can hand a function off from one operator to another, each taking the output from the previous and modifying it further. For example, D^2 differentiates twice: it is the second-derivative operator, sending $f(t)$ to $f''(t)$.

We have been studying ODEs of the form $m\ddot{x} + b\dot{x} + kx = q(t)$. The left hand side is the effect of an operator on the function $x(t)$, namely, the operator $mD^2 + bD + kI$. This *operator* describes the *system* (composed for example of a mass, dashpot, and spring).

We'll often denote an operator by a single capital letter, such as L. If $L = mD^2 + bD + kI$, for example, then our favorite ODE,

$$m\ddot{x} + b\dot{x} + kx = q$$

can be written simply as

$$Lx = q.$$

At this point m, b, and k could be functions of t.

Note well: the operator does NOT take the signal as input and return the system response, but rather the reverse: $Lx = q$, the operator takes the response and returns the signal. In a sense the system is better modeled by the "inverse" of the operator L. In rough terms, solving the ODE $Lx = q$ amounts to inverting the operator L.

Here are some definitions. A **differential operator** is one which is algebraically composed of D's and multiplication by functions. $mD^2 + bD + kI$ is an example of a second order differential operator. The **order** of a differential operator is the highest derivative appearing in it.

This example has another important feature: it is *linear*. An operator L is *linear* if

$$L(cf) = cLf \qquad \text{and} \qquad L(f + g) = Lf + Lg.$$

10.2. **LTI operators and exponential signals.** We will study almost exclusively linear differential operators. They are the operators of the form

$$L = a_n(t)D^n + a_{n-1}(t)D^{n-1} + \cdots + a_0(t)I.$$

The functions $a_0, \ldots, a_n$ are the **coefficients** of L.

In this course we focus on the case in which the coefficients are *constant*; each a_k is thus a *number*, and we can form the **characteristic polynomial** of the operator,

$$p(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0.$$

The operator is **Linear** and **Time Invariant**: an **LTI** operator. The original operator is obtained by formally replacing the indeterminate s here with the differentiation operator D, so we may write

$$L = p(D).$$

The characteristic polynomial completely determines the operator, and many properties of the operator are conveniently described in terms of its characteristic polynomial.

Here is a first example of the power of the operator notation. Let r be any constant. (You might as well get used to thinking of it as a possibly *complex* constant.) Then

$$De^{rt} = re^{rt}.$$

(A fancy expression for this is to say that r is an *eigenvalue* of the operator D, with corresponding *eigenfunction* e^{rt}.) Iterating this we find that

$$D^k e^{rt} = r^k e^{rt}.$$

We can put these equations together, for varying k, and evaluate a general LTI operator

$$p(D) = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_0 I$$

on e^{rt}. The operator D^k pulls r^k out as a factor, and when you add them all up you get the value of the polynomial $p(s)$ at $s = r$:

$$p(D)e^{rt} = p(r)e^{rt}. \qquad (1)$$

It is crucial here that the operator be time invariant: If the coefficients a_k are not constant, then they don't just pull outside the differentiation; you need to use the product rule instead, and the formulas become more complicated—see Section 12.

Multiplying it by $a/p(r)$ we find this:

Exponential Response Formula: A solution to

$$p(D)x = ae^{rt} \qquad (2)$$

is given by the

$$\boxed{x_p = a\frac{e^{rt}}{p(r)}} \qquad (3)$$

provided only that $p(r) \neq 0$.

The Exponential Response Formula ties together many different parts of this course. Since the most important signals are exponential, and the most important differential operators are LTI operators, this single formula solves most of the ODEs you are likely to face in your future.

The function x_p given by (3) is the *only* solution to (2) which is a multiple of an exponential function. If r has the misfortune to be a root of $p(s)$, so that $p(r) = 0$, then the formula (3) would give a zero in the denominator. The conclusion is that there are *no* solutions which are multiples of exponential functions. This is a "resonance" situation. In this case we can still find an explicit solution; see Section 12 for this.

Example 10.2.1. Let's solve

$$\ddot{x} + \dot{x} + x = 1 + 2e^t \,. \tag{4}$$

This is an inhomogeneous linear equation, so the general solution as the form $x_p + x_h$, where x_p is any particular solution and x_h is the general homogeneous solution. The characteristic polynomial is $p(s) = s^2 + s + 1$, so the equation is $p(D)x = 1 + 2e^t$. The input signal is a linear combination of 1 and e^t, so, again by superposition, if x_1 is a solution of $p(D)x = 1$ and x_2 is a solution of $p(D)x = e^t$, then a solution to (4) is given by $x_p = x_1 + 2x_2$.

The constant function 1 is exponential: $1 = e^{rt}$ with $r = 0$. Thus $p(D)x = 1$ has for solution $1/p(0) = 1$. This is easily checked! So take $x_1 = 1$.

Similarly, we can take $x_2 = e^t/p(1) = e^t/3$. Thus

$$x_p = 1 + 2e^t/3 \,.$$

Remark 10.2.2. The quantity

$$W(s) = \frac{1}{p(s)}$$

that occurs in the Exponential Signal Formula (3) is the **transfer function** of the system. One usually encounters this in the context of the Laplace transform, but it has a clear interpretation for us already: for any given r, one response of the system to the exponential signal e^{rt} is simply $W(r)e^{rt}$ (as long as $p(r) \neq 0$).

The transfer function is sometimes called the **system function** (e.g. by Oppenheim and Willsky) or the **complex gain**, and it is often written as $H(s)$.

10.3. **Sinusoidal signals: examples.** Being able to handle exponential signals is even more significant than you might think at first, because of the richness of the *complex* exponential. To exploit this richness, we have to allow complex valued functions of t. The main complex valued function we have to consider is the complex exponential function $z = e^{wt}$, where w is some fixed complex number. We know its derivative, by the Exponential Principle: $\dot{z} = we^{wt}$.

Here's how we can use this. Suppose we want to solve

$$\ddot{x} + \dot{x} + x = 2\cos(2t). \tag{5}$$

Step 1. Find a complex valued equation with an exponential signal of which this is the real (or, if the input signal is a sine, imaginary) part.

There are various ways to do this, but the most natural one is to view $2\cos(2t)$ as the real part of $2e^{2it}$ and write down

$$\ddot{z} + \dot{z} + z = 2e^{2it}\,. \tag{6}$$

This is a *new* equation, different from the original one. Its solution deserves a different name, and we have chosen one for it: z. This introduction of a new variable name is an essential part of Step 1. The real part of a solution to (6) is a solution to (5): $\operatorname{Re} z = x$.

Step 2. Find a particular solution z_p to the new equation.

By the Exponential Response Formula (3)

$$z_p = 2\frac{e^{2it}}{p(2i)}\,.$$

Compute:

$$p(2i) = (2i)^2 + 2i + 1 = -3 + 2i$$

so

$$z_p = 2\frac{e^{2it}}{-3+2i}\,. \tag{7}$$

Step 3. Extract the real (or imaginary) part of z_p to recover x_p. The result will be a sinusoidal function, and there are good ways to get to either expression for a sinusoidal function.

Rectangular version. Write out the real and imaginary parts of the exponential and rationalize the denominator:

$$z_p = 2\frac{(-3-2i)(\cos(2t) + i\sin(2t))}{9+4}\,.$$

The real part is

$$x_p = \frac{-6\cos(2t) + 4\sin(2t)}{13}\,, \tag{8}$$

and there is our solution!

Polar version. To obtain the "polar form" $x_p = A\cos(\omega t - \phi)$ studied in Section 4, you can use the expression (8) above and the relation from (3) in Section 4. But it's easier to proceed directly to the polar form from the expression (7) instead. To do this, write the factor $2/(-3+2i)$ in polar form:

$$\frac{2}{-3+2i} = ge^{-i\phi}\,,$$

so g is the magnitude and $-\phi$ is the angle. (We use $-\phi$ instead of ϕ is because we will want to wind up with a phase *lag*.) $g = 2/|-3+2i| =$

$2/\sqrt{13}$, and $-\phi$ is the argument of $1/(-3+2i)$ (so ϕ is the argument of $-3+2i$), which is approximately 2.55 radians or 146°. Then

$$z_p = ge^{-\phi i}e^{2it} = \frac{2}{\sqrt{13}}e^{(2t-\phi)i}\,.$$

The real part is now exactly

$$x_p = \frac{2}{\sqrt{13}}\cos(2t-\phi)\,.$$

Exercise 10.3.1. Carry out the same process to find a particular solution to $\ddot{x}+\dot{x}+x = (2+e^{-t})\cos(2t)$. (Hint: use superposition.)

Example 10.3.2. The harmonic oscillator with sinusoidal forcing term:

$$\ddot{x}+\omega_n^2 x = A\cos(\omega t)\,.$$

This is the real part of the equation

$$\ddot{z}+\omega_n^2 z = Ae^{i\omega t}\,,$$

which we can solve directly from the Exponential Response Formula: since $p(i\omega) = (i\omega)^2+\omega_n^2 = \omega_n^2-\omega^2$,

$$z_p = A\frac{e^{i\omega t}}{\omega_n^2-\omega^2}$$

as long as the input frequency is different from the natural frequency of the harmonic oscillator. Since the denominator is *real*, the real part of z_p is easy to find:

$$x_p = A\frac{\cos(\omega t)}{\omega_n^2-\omega^2}\,. \tag{9}$$

Similarly, the sinusoidal solution to

$$\ddot{y}+\omega_n^2 y = A\sin(\omega t)$$

is the imaginary part of z_p,

$$y_p = A\frac{\sin(\omega t)}{\omega_n^2-\omega^2}\,. \tag{10}$$

This solution puts in precise form some of the things we can check from experimentation with vibrating systems. When the frequency of the signal is smaller than the natural frequency of the system, $\omega < \omega_n$, the denominator is positive. The effect is that the system response is a *positive* multiple of the signal: the vibration of the mass is "in sync" with the impressed force. As ω increases towards ω_n, the denominator in (9) nears zero, so the amplitude of the solution grows arbitrarily large. This is **near resonance**. When $\omega = \omega_n$ the system is **in resonance** with the signal; the Exponential Response Formula fails, and there is *no* periodic (or even bounded) solution. (We'll see in Section 12 how to get a solution in this case.) When $\omega > \omega_n$, the denominator is negative. The system response is a *negative* multiple of the signal: the vibration of the mass is perfectly "out of sync" with the impressed force.

Since the coefficients are constant here, a time-shift of the signal results in the same time-shift of the solution:

$$\ddot{x} + \omega_n^2 x = A\cos(\omega t - \phi)$$

has the periodic solution

$$x_p = A\frac{\cos(\omega t - \phi)}{\omega_n^2 - \omega^2}.$$

The equations (9) and (10) will be very useful to us when we solve ODEs via Fourier series.

Exercise 10.3.3. Find a particular solution to $\ddot{x}+\dot{x}+x = \cos(2t-\pi/4)$.

10.4. **Sinusoidal signals: theory.** To generalize these examples, consider a general LTI system driven by a sinusoidal input signal:

$$p(D)x = a\cos(\omega t)\,. \tag{11}$$

Replace this by the complex equation of which it is the real part:

$$p(D)z = ae^{i\omega t}\,.$$

Unless $i\omega$ is a root of the characteristic polynomial $p(s)$, the Exponential Response Formula gives the solution

$$z_p = a\,\frac{1}{p(i\omega)}\,e^{i\omega t}\,.$$

Using the notation of Remark 10.2.2, the multiplier

$$\frac{1}{p(i\omega)} = W(i\omega)$$

is the transfer function evaluated at the imaginary number $i\omega$. This complex number $W(i\omega)$ is the **frequency response** or **complex gain** of the operator $p(D)$ or of the system that it represents. A sinusoidal solution to (11) can be obtained by finding the real part of z_p, and, as in the example above, it is easy to express the amplitude and phase lag of the real part of z_p by first writing down the polar expression for $W(i\omega)$ as

$$W(i\omega) = ge^{-i\phi}\,.$$

(Thus g and ϕ are both real valued functions of ω.) Then

$$z_p = cge^{-i\phi}e^{i\omega t} = cge^{i(\omega t-\phi)}\,,$$

and its real part is

$$x_p = cg\cos(\omega t - \phi)\,.$$

The magnitude of the complex gain,

$$g = |W(i\omega)|\,,$$

is the **gain**, and the negative of the argument,

$$\phi = -\arg W(i\omega)\,,$$

is the **phase lag**. In fact for any sinusoidal input $A\cos(\omega t - \phi_0)$ with circular frequency ω, the sinusoidal output is given by $gA\cos(\omega t - \phi_0 - \phi)$: magnified by the factor g and lagging behind the input by ϕ radians.