

11. UNDETERMINED COEFFICIENTS

In this section we describe now to solve the constant coefficient linear ODE $p(D)x = q(t)$ in case $q(t)$ is polynomial rather than exponential. Any function can be approximated in a suitable sense by polynomial functions, and this makes polynomials a flexible and important tool.

A *polynomial* is a function of the form

$$q(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_0.$$

The smallest k for which $a_k \neq 0$ is the *degree* of $q(t)$. The zero function is a polynomial too, but it doesn't have a degree.)

Note that $q(0) = a_0$ and $q'(0) = a_1$.

Here is the basic fact about the response of an LTI system with characteristic polynomial $p(s)$ to polynomial signals:

Theorem. (Undetermined coefficients) If $p(0) \neq 0$, and $q(t)$ is a polynomial of degree n , then

$$p(D)x = q(t)$$

has exactly one solution which is polynomial, and it is of degree n .

The best way to see this, and to see how to compute this polynomial particular solution, is by an example. Suppose we have

$$\ddot{x} + 2\dot{x} + 3x = 4t^2 + 5.$$

The theorem asserts that there is exactly one solution of the form

$$x = at^2 + bt + c,$$

where a, b, c are constants. To find them, just substitute this expression for x into the equation. It's helpful to be systematic in making this computation. Write out x, \dot{x} , and \ddot{x} , and then multiply by the coefficients, taking care to line up powers of t :

$$\begin{array}{rccccccc} 3x & = & 3at^2 & + & 3bt & + & 3c \\ 2\dot{x} & = & & & 4at & + & 2b \\ \ddot{x} & = & & & & & 2a \\ \hline 4t^2 + 5 & = & 3at^2 & + & (4a + 3b)t & + & (2a + 2b + 3c) \end{array}$$

Now we equate coefficients of corresponding powers of t . It's easiest to start with the highest power of t :

$$\begin{aligned} 4 &= 3a & \text{so} & & a &= 4/3, \\ 3b &= -4a = -16/3 & \text{so} & & b &= -16/9, \\ 3c &= 5 - 2(4/3) - 2(-16/9) & \text{so} & & c &= 53/27. \end{aligned}$$

Our solution is thus

$$x = (4/3)t^2 - (16/9)t + (53/27).$$

The computation in this example is about as complicated as it could get. I planned it that way so you would see the point: since it's an "upper triangular" set of linear equations, you can always solve for the coefficients one after the other.

If the constant term in the characteristic polynomial had been zero, though, there would have been trouble: there would have been nothing on the right and side of the table to give t^2 . This is why the hypothesis in the theorem is needed.

There is a simple dodge we can use in case $p(0) = 0$, though. If $p(0) = 0$, then $p(D)x$ doesn't involve x itself; it involves only \dot{x} and *its* derivatives. So we can regard it as an ODE (of one order less) for \dot{x} , solve that, and then integrate to solve for x . It may be useful to write $y = \dot{x}$ in this process, to keep your head straight.

Suppose we have

$$\ddot{x} + 2\dot{x} = 3t^2,$$

for example. If we write $y = \dot{x}$, this is the same as

$$\dot{y} + 2y = 3t^2.$$

The theorem applies to this equation, now: there is exactly one solution of the form $y = at^2 + bt + c$.

Exercise. Carry out the computation to show that this polynomial solution is $y = (3/2)t^2 - (3/2)t + 3/4$.

Now a solution to the original problem is given by an integral of y : $x = (1/2)t^3 - (3/4)t^2 + (3/4)t$. You still get a polynomial solution, but it is no longer the only polynomial solution—I can add any constant to it and get another—and its degree is larger than the degree of the input function.

These methods let you find a polynomial response to a polynomial signal for any LTI system.

Final remark: there is overlap with the case of exponential signal, since the constant function with value 1 is an exponential: $e^{0t} = 1$. Our earlier method gives the solution $e^{0t}/p(0)$ for a solution to $p(D)x = 1$, provided $p(0) \neq 0$. This the same as the solution given by the method of undetermined coefficients.