13. Natural frequency and damping ratio

We'll consider the second order homogeneous linear constant coefficient ODE

$$\ddot{x} + b\dot{x} + cx = 0$$

with positive "spring constant/mass" c. In the absence of a damping term this spring constant would be the square of the natural circular frequency of the system, so we will write it as ω_n^2 with $\omega_n > 0$, and call ω_n the **natural circular frequency** of the system.

Critical damping occurs when the coefficient of $\dot{x}$ is $2\omega_n$. The **damping ratio** ζ is the ratio of b to the critical damping constant: $\zeta = b/2\omega_n$. The ODE then has the form

$$\boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0} \tag{1}$$

Note that if x has dimensions of cm and t of sec, then ω_n had dimensions $\sec^{-1}$, and the damping ratio ζ is "dimensionless," a number which is the same no matter what units of distance or time are chosen. Critical damping occurs precisely when $\zeta = 1$: then the characteristic polynomial has a repeated root: $p(s) = (s + \omega_n)^2$.

In general the characteristic polynomial is $s^2 + 2\zeta\omega_n s + \omega_n^2$, and it has as roots

$$-\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2} = \omega_n(-\zeta \pm \sqrt{\zeta^2 - 1}).$$

These are real when $|\zeta| \geq 1$, equal when $\zeta = \pm 1$, and nonreal when $|\zeta| < 1$. When $|\zeta| \leq 1$, the roots are

$$\omega_n(-\zeta \pm i\sqrt{1 - \zeta^2}).$$

These are complex numbers of magnitude ω_n and argument $\pm\theta$, where $-\zeta = \cos\theta$.

Suppose we have such a system, but don't know the values of ω_n or ζ. At least when the system is underdamped, we can discover them by a simple experiment. Let's displace the mass and watch it vibrate freely. In the underdamped case, the general solution of the homogeneous equation is

$$x = Ae^{-\zeta\omega_n t}\cos(\omega_d t - \phi) \tag{2}$$

where

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} \tag{3}$$

is the **damped circular frequency** of the system. Notice the effect of damping on the circular frequency! It decreases from its undamped ("natural") value by a factor of $\sqrt{1-\zeta^2}$.

Let's study the times at which x achieves its maxima. These occur when the derivative vanishes, and

$$\dot{x} = Ae^{-\zeta\omega_n t}\left(-\zeta\omega_n \cos(\omega_d t - \phi) - \omega_d \sin(\omega_d t - \phi)\right).$$

The factor in parentheses is sinusoidal with circular frequency ω_d, so successive zeros are separated from each other by a time lapse of π/ω_d. If t_1 and t_2 are the times of neighboring maxima of x (which occur at every other extremum) then $t_2 - t_1 = 2\pi/\omega_d$, so we have discovered the damped natural frequency:

$$\omega_d = \frac{2\pi}{t_2 - t_1}. \tag{4}$$

We can also measure the ratio of the value of x at two successive maxima. Write $x_1 = x(t_1)$ and $x_2 = x(t_2)$. The difference of their natural logarithms is the **logarithmic decrement**:

$$\Delta = \ln x_1 - \ln x_2 = \ln\left(\frac{x_1}{x_2}\right).$$

Then

$$x_2 = e^{-\Delta} x_1.$$

The logarithmic decrement turns out to depend only on the damping ratio. To see this, note that the values of $\cos(\omega_d t - \phi)$ at two points of time differing by $2\pi/\omega_d$ are equal. Using (2) we find

$$\frac{x_1}{x_2} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n t_2}} = e^{\zeta\omega_n(t_2 - t_1)}.$$

Thus, using (4) and (3),

$$\Delta = \ln\left(\frac{x_1}{x_2}\right) = \zeta\omega_n(t_2 - t_1) = \zeta\omega_n \frac{2\pi}{\omega_d} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}.$$

From the quantities ω_d and Δ, which are directly measurable characteristics of the unforced system response, we can calculate the system parameters ω_n and ζ:

$$\zeta = \frac{\Delta/2\pi}{\sqrt{1 + (\Delta/2\pi)^2}}, \qquad \omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = \sqrt{1 + \left(\frac{\Delta}{2\pi}\right)^2}\,\omega_d\,. \tag{5}$$