14. Frequency response

In Section 3 we studied the frequency response of a first order LTI operator. In Section 10.4 we used the Exponential Response Formula to understand the response of a general LTI operator to a sinusoidal input signal. Here we will study this in more detail in case the operator is of second order, and understand how the gain and phase lag vary with the driving frequency. We will add a note about the use of an LTI system as a "filter."

14.1. **Second order frequency response.**

We are looking at a second order LTI ODE with a sinusoidal driving force. We may as well set the clock so that the force is maximal at $t = 0$. By dividing by the coefficient of $\ddot{x}$ we obtain the differential equation

$$\ddot{x} + b\dot{x} + cx = a\cos(\omega t). \tag{1}$$

The parameters have the following significance: b is the damping constant, c is the spring constant, a is the amplitude of the signal (all divided by the mass), and ω is the circular frequency of the signal. We'll assume that all these are nonnegative.

As explained in Section 13, it is useful to write $c = \omega_n^2$ (where we choose $\omega_n \geq 0$) and $b = 2\zeta\omega_n$; ω_n is the "undamped natural circular frequency," and ζ is the "damping ratio."

The best path to the solution of (1) is to view it as the real part of the complex equation

$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = ae^{i\omega t}. \tag{2}$$

The Exponential Response Formula of Section 10 tells us that unless $\zeta = 0$ and $\omega = \omega_n$ (in which case the equation exhibits resonance, and has no periodic solutions), this has the particular solution

$$z_p = a\,\frac{e^{i\omega t}}{p(i\omega)} \tag{3}$$

where $p(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$ is the characteristic polynomial of the system. In Section 10 we wrote $W(s) = 1/p(s)$, so this solution can be written

$$z_p = aW(i\omega)e^{i\omega t}\,.$$

The complex valued function of ω given by $W(i\omega)$ is the **frequency response** or the **complex gain**. We will see now how, for fixed ω, this function contains exactly what is needed to write down a sinusoidal solution to (1).

As in Section 10.4 we can go directly to the expression in terms of amplitude and phase lag for the particular solution to (1) given by the real part of z_p as follows. Write the polar expression (as in Section 6) for the complex number $W(i\omega) = 1/p(i\omega)$ as

$$\frac{1}{p(i\omega)} = ge^{-i\phi}. \tag{4}$$

(The use of $-\phi$ rather than ϕ is a little awkward, but it's forced on us by our choice to use the symbol ϕ for the phase *lag* rather than the phase *gain*, as we will see just below.) Then

$$z_p = age^{i(\omega t-\phi)}, \qquad x_p = ag\cos(\omega t - \phi),$$

g is the "amplitude gain" or just "gain," and ϕ is the "phase lag" (so $-\phi$ is the "phase gain" or "phase shift"). This is the *only* periodic solution to (1), and, assuming $\zeta > 0$, any other solution differs from it by a transient. This solution is therefore the most important one. Its graph is easy to reconstruct and visualize from the amplitude and phase lag, and so we will focus on those two numbers and leave it up to you and your imagination to reconstruct the explicit sinusoidal solution they determine. We want to understand how g and ϕ depend upon the driving frequency ω.

A short computation shows that the gain g is given in terms of ω by

$$g(\omega) = \frac{1}{|p(i\omega)|} = \frac{1}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}}. \tag{5}$$

We can watch what happens to the system response as the signal is tuned to different frequencies. At the extremes: (1) $g(0) = 1/\omega_n^2$ and $g'(0) = 0$, so when ω is small—so the period of the signal is large—$g(\omega)$ is approximately the constant $1/\omega_n^2$. (2) When ω is large relative to ω_n, $g(\omega)$ is approximately $1/\omega^2$.

Figure 6 shows the graphs of gain against the circular frequency of the signal for $\omega_n = 1$ and several values of the damping ratio ζ (namely $\zeta = 1/(4\sqrt{2}), 1/4, 1/(2\sqrt{2}), 1/2, 1/\sqrt{2}, 1, \sqrt{2}, 2$.) As you can see, the gain may achieve a maximum. This occurs when the square of the denominator in (5) is minimal, and we can discover where this is by differentiating with respect to ω and setting the result equal to zero:

$$\frac{d}{d\omega}\left((\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2\right) = -2(\omega_n^2 - \omega^2)2\omega + 8\zeta^2\omega_n^2\omega, \tag{6}$$

and this becomes zero when ω takes on the value

$$\omega_r = \omega_n\sqrt{1 - 2\zeta^2}. \tag{7}$$

When $\zeta = 0$ the gain becomes infinite at $\omega = \omega_n$: this is resonance. As ζ increases from zero, the maximal gain of the system occurs at smaller and smaller frequencies, till when $\zeta > 1/\sqrt{2}$ no such maximum occurs. The occurrence of a peak gain is called **practical resonance**.

We also have the phase lag to consider: the periodic solution to (1) is

$$x_p = g\cos(\omega t - \phi).$$

Returning to (4), ϕ is given by the argument of the complex number

$$p(i\omega) = (\omega_n^2 - \omega^2) + 2i\zeta\omega_n\omega.$$

This is the angle counterclockwise from the positive x axis of the ray through the point $(\omega_n^2 - \omega^2, 2\zeta\omega_n\omega)$. Since ζ and ω are nonnegative, this point is always in the upper half plane, and $0 \le \phi \le \pi$. The phase response graphs for $\omega_n = 1$ and several values of ζ are shown in the second figure.

When $\omega = 0$, there is no phase lag, and when ω is small, ϕ is approximately $2\zeta\omega/\omega_n$. $\phi = \pi/2$ when $\omega = \omega_n$, independent of the damping rato ζ: when the signal is tuned to the natural frequency of the system, the phase lag is $\pi/2$, which is to say that the time lag is one-quarter of a period. As ω gets large, the phase lag tends towards π: strange as it may seem, the sign of the system response tends to be opposite to the sign of the signal.

Engineers also typically have to deal with a very wide range of frequencies. In order to accommodate this, and to show the behavior of the frequency response more clearly, they tend to plot $\log_{10}|1/p(i\omega)|$ and the argument of $1/p(i\omega)$ against $\log_{10}\omega$. These are the so-called **Bode plots**.

The expression $1/p(i\omega)$, as a complex-valued function of ω, contains complete information about the system response to periodic input signals. If you let ω run from $-\infty$ to ∞ you get a curve in the complex plane called the **Nyquist plot**. In cases that concern us we may restrict attention to the portion parametrized by $\omega > 0$. For one thing, the characteristic polynomial $p(s)$ has real coefficients, which means that $p(-i\omega) = p(\overline{i\omega}) = \overline{p(i\omega)}$ and so $1/p(-i\omega)$ is the complex conjugate of $1/p(i\omega)$. The curve parametrized by $\omega < 0$ is thus the reflection of the curve parametrized by $\omega > 0$ across the real axis.

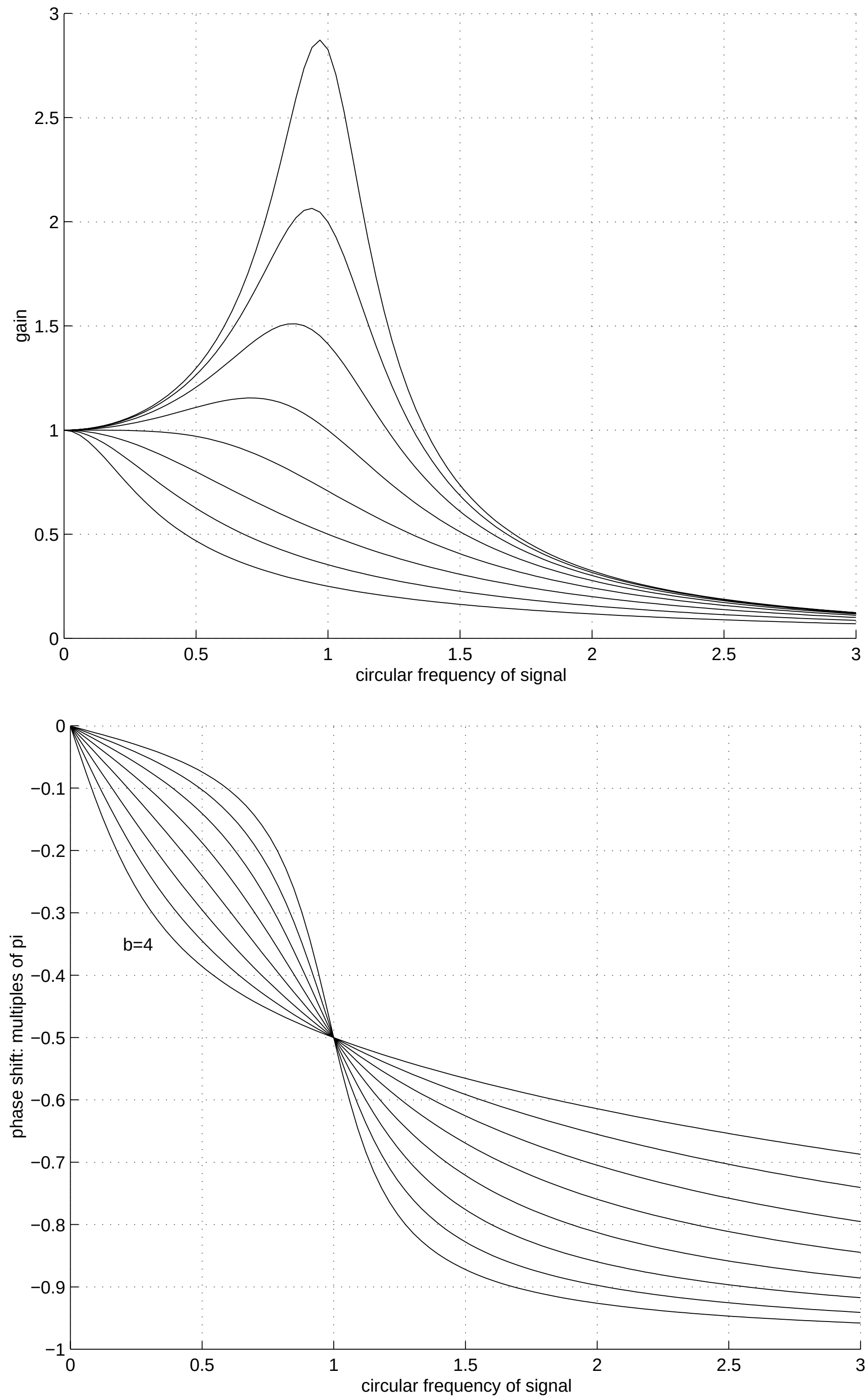

FIGURE 6. Second order amplitude response curves

14.2. **Filters.** Understanding the frequency response allows engineers to use these systems as **filters**. Ideally, one may wish to build a system with a fixed positive gain for all circular frequencies smaller than some critical value ω_c, and gain zero for larger circular frequencies. This is called a "low-pass filter": it lets low frequencies through in a uniform way but is opaque to higher frequencies. No actual system can match this perfectly, and certainly not ones modeled by second order equations. Nevertheless, engineering like politics is the art of the possible, and one attempts to do the best one can. How you measure "the best" depends upon circumstances. One popular choice is to ask for the flattest possible gain graph for small values of ω, and accept whatever follows for larger values of ω.

The shape of the graph of $g(\omega)$ depends upon the parameters ω_n and ζ. Imagine fixing ω_n and varying ζ to obtain the flattest possible amplitude response at $\omega = 0$. In (6) we computed the numerator of $g'(\omega)$, and the factor of ω shows that $g'(0) = 0$ for any ζ. To get "flatter" than that we hope that the *parabola* best approximating $g(\omega)$ at $\omega = 0$ is a horizontal straight line; that is, we look for ζ such that $g''(\omega) = 0$. A little thought shows that this occurs when practical resonance occurs at $\omega = 0$, that is, when $\omega_r = 0$. By (7) this means

$$\zeta = 1/\sqrt{2}.$$

In this case, the gain function has the particularly simple form

$$g(\omega) = \frac{1}{\sqrt{\omega_n^4 + \omega^4}}$$

This system, modeled by the operator $L = D^2 + \sqrt{2}\,\omega_n D + \omega_n^2 I$, is the *second order Butterworth filter*. Its gain graph, in case $\omega_n = 1$, is among those plotted in the figure. There are analogous filters of higher order, which exhibit sharper frequency cut-offs.

Another attractive feature of the Butterworth filter is that its phase response is close to linear for ω not too big. Phase shift often presents engineering problems of its own. It is unavoidable, but at least if it's linear in the frequency of the signal it's easier to deal with.