15. The Wronskian

We know that a general second order homogeneous linear ODE,

$$y'' + p(x)y' + q(x)y = 0, \tag{1}$$

has a pair of independent solutions; and that if y_1, y_2 is any pair of independent solutions then the general solution is

$$y = c_1 y_1 + c_2 y_2. \tag{2}$$

Suppose we wish to solve the initial value problem with

$$y(x_0) = a\,, \quad y'(x_0) = b.$$

To solve for the constants c_1 and c_2 in (2), we get one equation by substituting $x = x_0$ in to this expression. We get another equation by first differentiating (2) and then setting $x = x_0$ and using the value $y'(x_0) = b$. We end up with the system of linear equations

$$y_1(x_0)c_1 + y_2(x_0)c_2 = a\,, \quad y_1'(x_0)c_1 + y_2'(x_0)c_2 = b \tag{3}$$

We can write down the solutions uniformly in the coefficients of these equations:

$$c_1 = \frac{y_2'(x_0)a - y_2(x_0)b}{W(x_0)}\,, \quad c_2 = \frac{-y_1'(x_0)a + y_1(x_0)b}{W(x_0)}$$

where $W(x_0)$ is the value at x_0 of the **Wronskian** function

$$W(x) = y_1 y_2' - y_2 y_1' = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \tag{4}$$

determined by the pair of solutions y_1, y_2.

You generally wouldn't want to use these formulas for the coefficients; it's better to compute them directly from (3) in the particular case you are looking at. But this calculation does draw attention to the Wronskian function. We can find a linear combination of y_1 and y_2 which solves the IVP for any given choice of initial conditions exactly when $W(x_0) \neq 0$.

On the other hand it's a theorem that one can solve the initial value problem at *any* x value using a linear combination of any linearly independent pair of solutions. A little thought then leads to the following conclusion:

Theorem. Let y_1, y_2 be solutions of (1) and let W be the Wronskian formed from y_1, y_2. Either W is the zero function and one solution is a multiple of the other, or the Wronskian is nowhere zero, neither solution is a multiple of the other, and any solution is a linear combination of y_1, y_2.

For example, if we compute the Wronskian of the pair of solutions $\{\cos x, \sin x\}$ of $y'' + y = 0$, we get the constant function 1, while the Wronskian of $\{\cos x, 2\cos x\}$ is the constant function 0. One can show (as most ODE textbooks do) that if W is the Wronskian of *some* linearly independent pair of solutions, then the Wronskian of *any* pair of solutions is a constant multiple of W. (That multiple is zero if the new pair happens to be linearly dependent.)

Many references, including Edwards and Penney, encourage the impression that computing the Wronskian of a pair of functions is a good way to check whether or not they are linearly independent. This is silly. Two functions are linearly dependent if one is a multiple of the other; otherwise they are linearly independent. This is always easy to see by inspection.

Nevertheless the Wronskian can teach us important things. To illustrate one, let's consider an example of a second order linear homogeneous system with *nonconstant* coefficient: the **Airy equation**

$$y'' + xy = 0. \tag{5}$$

At least for $x > 0$, this is like the harmonic oscillator $y'' + \omega_n^2 y = 0$, except that the natural circular frequency ω_n keeps increasing with x: the x sits in the position where we expect to see ω_n^2, so near to a given value of x we expect solutions to behave like $\cos(\sqrt{x}x)$ and $\sin(\sqrt{x}x)$. I emphasize that these functions are *not* solutions to (5), but they give us a hint of what to expect. In fact the normalized pair (see Section 9) of solutions to (5), the "Airy cosine and sine functions," have graphs as illustrated in Figure 7

One of the features this picture has in common with the graphs of cosine and sine is the following fact, which we state as a theorem.

Theorem. Let $\{y_1, y_2\}$ be any linearly independent pair of solutions of the second order linear ODE (1), and suppose that x_0 and x_1 are numbers such that $x_0 \neq x_1$ and $y_1(x_0) = 0 = y_1(x_1)$. Then y_2 becomes zero somewhere between x_0 and x_1.

This fact, that zeros of independent solutions interleave, is thus a completely general feature of second order linear equations. It doesn't depend upon the solutions being normalized, and it doesn't depend upon the coefficients being constant.

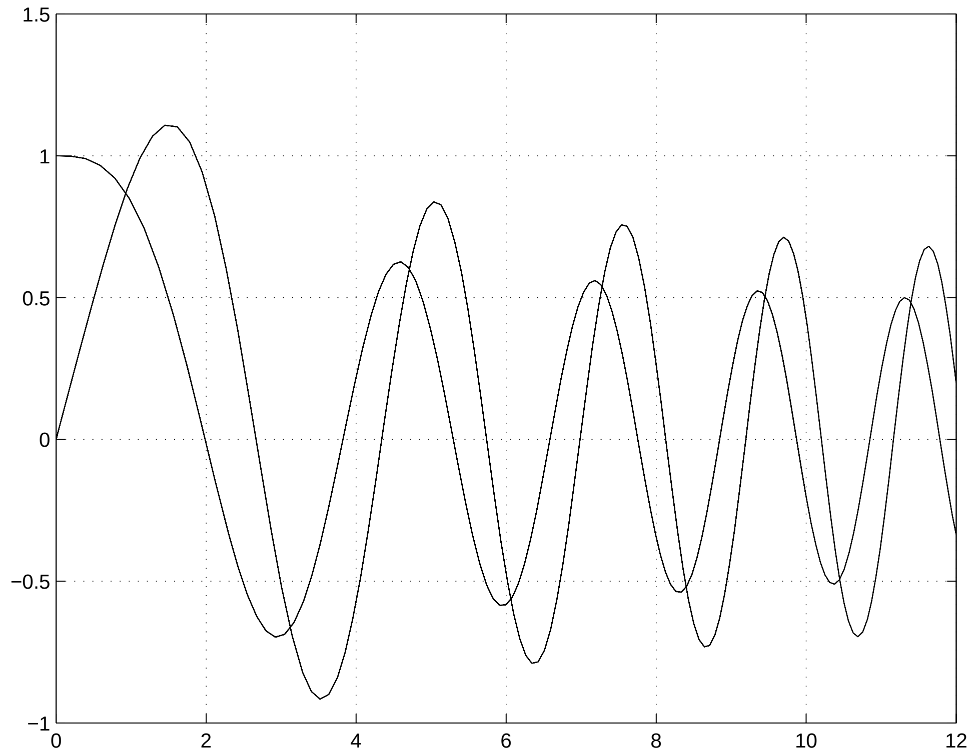

FIGURE 7. Airy cosine and sine

You can see why this must be true using the Wronskian. We might as well assume that y_1 is not zero anywhere between x_0 and x_1. Since the two solutions are independent the associated Wronskian is nowhere zero, and thus has the same sign everywhere. Suppose first that the sign is positive. Then $y_1 y_2' > y_1' y_2$ everywhere. At x_0 this says that $y_1'(x_0)$ and $y_2(x_0)$ have opposite signs, since $y_1(x_0) = 0$. Similarly, $y_1'(x_1)$ and $y_2(x_1)$ have opposite signs. But $y_1'(x_0)$ and $y_1'(x_1)$ must have opposite signs as well, since x_0 and x_1 are neighboring zeros of y_1. (These derivatives can't be zero, since if they were both terms in the definition of the Wronskian would be zero, but $W(x_0)$ and $W(x_1)$ are nonzero.) It follows that $y_2(x_0)$ and $y_2(x_1)$ have opposite signs, and so y_2 must vanish somewhere in between. The argument is very similar if the sign of the Wronskian is negative.