

17. IMPULSES AND GENERALIZED FUNCTIONS

In calculus you learn how to model processes using functions. Functions have their limitations, though, and, at least as they are treated in calculus, they are not convenient for modeling some important processes and events, especially those involving sudden changes. In this section we explain how the function concept can be extended to a wider class of objects which conveniently model such processes.

17.1. From bank accounts to the delta function. Recall from Section 2 the model of the savings account, beginning with the “difference equation”

$$(1) \quad x(t + \Delta t) = x(t) + I(t)x(t)\Delta t + q(t)\Delta t$$

and continuing to the differential equation

$$(2) \quad \dot{x} - I(t)x = q(t).$$

Here $I(t)$ is the interest rate at time t and $q(t)$ is the rate of contribution. Here $q(t)$ is measured in dollars per year, and, being a mathematician, I have taken the limit and replaced frequent small payments (say \$1 each day) by a continual payment (at a constant *rate* of $q(t) = 365$ dollars per year).

In fact, banks do not behave like mathematicians and take this limit. They use the difference equation (1) and compute intensively. Solving the ODE (2) is much simpler, and leads to good agreement with the discrete calculation. This continuous approximation is critical to the use of differential equations in modeling. The world is in fact discrete, made up of atoms (and smaller discrete structures), but in order to understand this seething digital mass we make continuous—even differentiable—approximations.

On the other hand, there are some processes which cannot be conveniently accommodated by this paradigm. For example, while I continue to save at the rate of \$1/day, my wealthy aunt decided to give me \$1000 as a birthday present, and I deposited this into the bank in one lump sum. How can we model this process?

One way is the following: solve the ODE (2) with $q = 365$, reflecting the compounding of interest in my account, subject to an appropriate initial condition, say $x(0) = x_0$. Then, at the moment I plan to deposit the gift, say at $t = t_1 > 0$, stop this process and compute my balance at the instant of the big deposit, $x(t_1)$. Write x_1 for this number. Then start the ODE up again with a new initial condition, $x(t_1) = x_1 + 1000$,

and solve anew to get $x(t)$ for $t > t_1$. Another way to think of this is to imagine that a separate fund is set up with the \$1000 gift, and allowed to grow at the same interest rate. The two perspectives are equivalent, by superposition. We get

$$x(t) = \begin{cases} -q/I + (x_0 + q/I)e^{It} & \text{if } 0 < t < t_1 \\ -q/I + (x_0 + q/I)e^{It} + 1000e^{I(t-t_1)} & \text{if } t > t_1 \end{cases}$$

The number $t - t_1$ is the amount of time the gift has been in the bank at time t .

I'd like to introduce some notation here. My bank balance seems to have *two* values at $t = t_1$: x_1 , and $x_1 + 1000$. There is notation to handle this. One writes $x(t_1-)$ for the balance at $t = t_1$ computed from times before the gift; mathematically,

$$x(t_1-) = \lim_{t \uparrow t_1} x(t).$$

Similarly, $x(t_1+)$ is the balance at $t = t_1$ from the perspective of later times; mathematically,

$$x(t_1+) = \lim_{t \downarrow t_1} x(t).$$

The actual value we assign as $x(t_1)$ is unimportant and can be left undeclared.

The fact is that the approach we just used to deal with this windfall situation is often the preferred strategy. Still, it would be convenient to find some way to *incorporate* a one-time essentially instantaneous gift into the *rate* $q(t)$. Then there's the withdrawal I made when I bought the BMW, too—so the machinery should accommodate a series of such events, at different times, of various magnitudes, and of either sign.

A good way to understand a rate is by considering the cumulative total. The cumulative total contribution to the account, from time zero up to time t , is

$$Q(t) = \int_0^t q(\tau) d\tau,$$

so in our case (before the gift) $Q(t) = 365t$. (Note that I had to come up with a new symbol for the time variable inside the integral.) Then the rate is given (by the fundamental theorem of calculus) as the derivative of the cumulative total: $q = Q'(t)$. When I incorporate my aunt's gift, the cumulative total jumps by \$1000 at time t_1 : so it is given by

$$Q(t) = \begin{cases} 365t & \text{for } t < t_1 \\ 365t + 1000 & \text{for } t > t_1 \end{cases}$$

(and it doesn't much matter what we declare $Q(t_1)$ to be.)

This is a perfectly good function, but it fails to be differentiable (in the usual sense) at $t = t_1$, so the corresponding rate, $q(t) = Q'(t)$, has problems at $t = t_1$.

We can approximate such a rate in the following way. Imagine that the gift isn't deposited all at once but instead over a short period of time, say h , starting at time t_1 . Thus the new rate function has a graph which is horizontal with value 365 until $t = t_1$, then it jumps to a value of $365 + 1000/h$, and then, at $t = t_1 + h$, it falls back to the constant value 365. We can try to take a limit here, letting $h \rightarrow 0$, but the result is not a function in the usual sense of the word.

Nevertheless, all these considerations indicate that there may be a mathematical concept a little more general than the function concept which serves to model a *rate which produces a jump in the cumulative total*. This is indeed the case: they are called *generalized functions*, and they are heavily studied by mathematicians and used by engineers and scientists. They form a convenient language.

17.2. The delta function. The most basic rate of this sort is the one which produces a "unit step" cumulative total:

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}$$

This $u(t)$ is an important function, and it is sometimes called the Heaviside function. It doesn't much matter what we declare $u(0)$ to be, and we'll just leave it unspecified; but we do know $u(0-) = 0$ and $u(0+) = 1$. The corresponding rate is the **Dirac delta function**,

$$\delta(t) = u'(t).$$

This object $\delta(t)$ behaves like an ordinary function, in fact like the constant function with value 0, except at $t = 0$, where it can be thought of as taking on such a large value that the area under the graph is 1. The delta function is also called the **unit impulse function**.

Using this notation, my rate of contribution, including my aunt's gift, is

$$q(t) = 365 + 1000 \delta(t - t_1).$$

Just as for (ordinary) functions, subtracting t_1 inside shifts the graph right by t_1 units; the spike occurs at $t = t_1$ rather than at $t = 0$.

We can also use the delta function to model discrete monthly contributions accurately, without replacing them with a continuous contribution. If the rate is \$365 per year, and I contribute at the start of each month (or more precisely at $t = 0, 1/12, 2/12, \dots$) starting at $t = 0$, then the rate is

$$q(t) = \frac{365}{12} \left(\delta(t) + \delta\left(t - \frac{1}{12}\right) + \delta\left(t - \frac{2}{12}\right) + \dots \right).$$

When these shifted and scaled delta functions are added to “ordinary” functions you get a “generalized function.” I’ll describe a little part of the theory of generalized functions. The next few paragraphs will sound technical. I hope they don’t obscure the simplicity of the idea of generalized functions as a model for abrupt changes.

I will use the following extensions of a definition from Edwards and Penney (p. 268): To prepare for it let me call a collection of real numbers a_1, a_2, \dots , **sparse** if for any $r > 0$ there are only finitely many of k such that $|a_k| < r$. So any finite collection of numbers is sparse; the collection of whole numbers is sparse; but the collection of numbers $1, 1/2, 1/3, \dots$, is not sparse. Sparse sets don’t bunch up. The empty set is sparse.

When I describe a function (on an interval) I typically won’t insist on knowing its values for *all* points in the interval. I’ll allow a sparse collection of points at which the value is undefined. We already saw this in the definition of $u(t)$ above.

A function $f(t)$ (on an interval) is **piecewise continuous** if (1) it is continuous everywhere (in its interval of definition) except at a sparse collection of points; and (2) for every a , both $f(a+)$ and $f(a-)$ exist.

A function $f(t)$ is **piecewise differentiable** if (1) it is piecewise continuous, (2) it is differentiable everywhere except at a sparse collection of points, and its derivative is piecewise continuous.

We now want to extend this by including delta functions. A **generalized function** is a piecewise continuous function $f_r(t)$ plus a linear combination of delta functions,

$$(3) \quad f_s(t) = \sum b_k \delta(t - a_k),$$

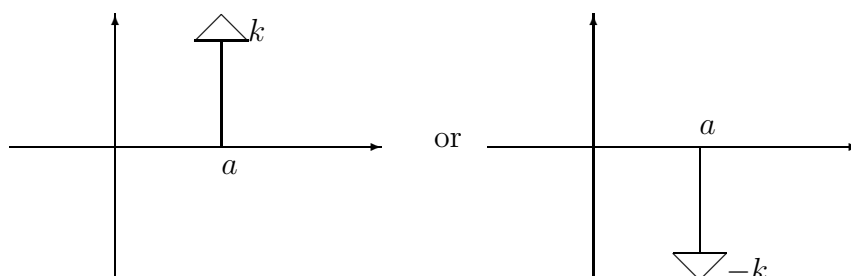
where the a_k ’s form a sparse set.

Write $f(t)$ for the sum:

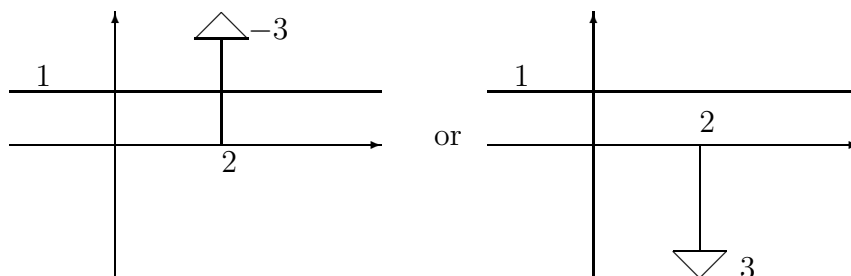
$$f(t) = f_r(t) + f_s(t).$$

$f_r(t)$ is the “regular part” of $f(t)$, and $f_s(t)$ is the “singular part.” We define $f(a-)$ to be $f_r(a-)$ and $f(a+)$ to be $f_r(a+)$. Since the actual value of $f(t)$ at $t = a$ is not used to compute these limits, this is a good definition even if a is one of the a_k 's.

We will use a “harpoon” to denote a delta function in a graph. The harpoon should be thought to be very high. This notation by itself does not include the information of the area under the graph. To deal with this we will decorate the barb of the harpoon representing $k\delta(t - a)$ with the number k . k may be negative, in which case the harpoon might better be thought of as extending downward. We will denote the same function, $k\delta(t - a)$ equally by a downward harpoon decorated with $-k$:



For example, $1 - 3\delta(t - 2)$ can be denoted by either of the following graphs.



A harpoon with $k = 0$ is the same thing as no harpoon at all: $0\delta(t - a) = 0$. We'll call the term $b_k\delta(t - a_k)$ occurring in $f_s(t)$ **the singularity of $f(t)$ at $t = a_k$** . If a is not among the a_k 's (or if $a = a_k$ but $b_k = 0$) then there is no singularity in $f(t)$ at $t = a$.

17.3. Integrating generalized functions. Generalized functions are set up so they can be integrated. We know what the integral of a delta function should be, since we are to think of it as the derivative of the unit step function:

$$\int_b^c \delta(t - a) dt = u(c - a) - u(b - a).$$

If $b < a < c$, this is 1. If a is not between b and c , this is 0. If $a = b$ or $a = c$ then this integral involves the expression $u(0)$, which is best thought of as undefined. We can however define

$$\int_{b-}^{c+} f(t) dt = \lim_{b' \uparrow b} \lim_{c' \downarrow c} \int_{b'}^{c'} f(t) dt,$$

and this gives a well defined result when $f(t) = \delta(t - a)$: Assuming $b \leq c$,

$$\int_{b-}^{c+} \delta(t - a) dt = 1 \quad \text{if } b \leq a \leq c,$$

and zero otherwise. In particular,

$$\int_{a-}^{a+} \delta(t - a) dt = 1.$$

Now if $f(t)$ is any generalized function, we can define the integral

$$\int_{b-}^{c+} f(t) dt$$

by integrating the regular part $f(t)$ in the usual way, and adding the sum of the b_k 's over k for which $b \leq a_k \leq c$ (using the notation of (3)).

The multiple of the delta function that occurs at $t = a$ in a generalized function can be expressed as

$$b = \int_{a-}^{a+} f(t) dt.$$

17.4. The generalized derivative. Generalized functions let us make sense of the derivative of a function which is merely piecewise differentiable.

For example, we began by saying that the “derivative” of the piecewise differentiable function $u(t - a)$ is the generalized function $\delta(t - a)$. This understanding lets us define the **generalized derivative** of any piecewise continuously differentiable function $f(t)$. It is a generalized function. Its regular part, $f'_r(t)$, is the usual derivative of $f(t)$ (which

is defined except where the graph of $f(t)$ has breaks or corners), and its singular part is given by the sum of terms

$$(f(a+) - f(a-))\delta(t - a),$$

summed over the values a of t where the graph of $f(t)$ has breaks. Each shifted and scaled δ function records the instantaneous velocity needed to accomplish a sudden jump in the value of $f(t)$. When the graph of $f(t)$ has a corner at $t = a$, the graph of $f'(t)$ has a jump at $t = a$ and isn't defined at $t = a$ itself; this is a discontinuity in the piecewise continuous function $f'_r(t)$.

With this definition, the “fundamental theorem of calculus”

$$\int_{b-}^{c+} f'(t) dt = f(c+) - f(b-)$$

holds for generalized functions.