21. THE LAPLACE TRANSFORM AND GENERALIZED FUNCTIONS

21.1. **Laplace transform of impulse and step responses.** Laplace transform affords a way to solve LTI IVPs. If the ODE is

$$p(D)x = f(t),$$

application of the Laplace transform results in an equation of the form

$$p(s)X = F(s) + G(s)$$

where $G(s)$ is computed from the initial conditions. Rest initial conditions lead to $G(s) = 0$, so in that case

$$X = W(s)F(s)$$

where $W(s) = 1/p(s)$ is the **transfer function** of the operator.

The very simplest case of this is when $f(t) = \delta(t)$. Then we are speaking of the unit impulse response $w(t)$, and we see that

> The Laplace transform of the unit impulse response $w(t)$ is the transfer function $W(s)$.

This is an efficient way to compute the unit impulse response.

The next simplest case is when $f(t) = u(t)$, the unit step function. Its Laplace transform is $1/s$, so the unit step response $w_1(t)$ is the inverse Laplace transform of

$$W_1(s) = \frac{W(s)}{s} = \frac{1}{sp(s)}.$$

By way of example, suppose the operator is $D^2 + 2D + 2$. The transfer function is $W(s) = 1/(s^2 + 2s + 2) = 1/((s+1)^2 + 1)$. By the s shift rule and the tables,

$$w(t) = u(t)e^{-t}\sin t.$$

The Laplace transform of the unit step response is $W_1(s) = 1/s(s^2 + 2s + 2)$, which we can handle using complex cover up: write

$$\frac{1}{s((s+1)^2 + 1)} = \frac{a}{s} + \frac{b(s+1) + c}{(s+1)^2 + 1}.$$

Multiply through by s and set $s = 0$ to see $a = 1/2$. Then multiply through by $(s+1)^2 + 1$ and set $s = -1 + i$ to see $bi + c = 1/(-1+i) = (-1-i)/2$, or $b = c = -1/2$: so

$$W_1(s) = \frac{1}{2}\left(\frac{1}{s} - \frac{(s+1)+1}{(s+1)^2 + 1}\right).$$

Thus the unit step response is

$$w_1(t) = \frac{u(t)}{2}(1 - e^{-t}(\cos t + \sin t))\,.$$

21.2. **What the Laplace transform doesn't tell us.** What do we mean, in the list of properties at the end of Section 20, when we say that $F(s)$ "essentially determines" $f(t)$?

The Laplace transform is defined by means of an integral. We don't need complete information about a function to determine its integral, so knowing its integral or integrals of products of it with exponentials won't be enough to completely determine it.

For example, if we can integrate a function $g(t)$ then we can also integrate any function which agrees with $g(t)$ except at one value of t, or even except at a finite number of values, and the integral of the new function is the same as the integral of g. Changing a few values doesn't change the "area under the graph."

Thus if $f(t) \rightsquigarrow F(s)$, and $g(t)$ coincides with $f(t)$ except at a few values of t, then also $g(t) \rightsquigarrow F(s)$. We can't hope to recover every value of $f(t)$ from $F(s)$ unless we put some side conditions on $f(t)$, such as requiring that it should be continuous.

Therefore, in working with functions via Laplace transform, when we talk about a function $f(t)$ it is often not meaningful to speak of the value of f at any specific point $t = a$. It does make sense to talk about $f(a-)$ and $f(a+)$, however. Recall that these are defined as

$$f(a-) = \lim_{t \uparrow a} f(t), \qquad f(a+) = \lim_{t \downarrow a} f(t).$$

This means that $f(a-)$ is the limiting value of $f(t)$ as t increases towards a from below, and $f(a+)$ is the limiting value of $f(t)$ as t decreases towards a from above. In both cases, the limit polls infinitely many values of f near a, and isn't changed by altering any finite number of them or by altering $f(a)$ itself; in fact f does not even need to be defined at a for us to be speak of $f(a\pm)$. The best policy is to speak of $f(a)$ only in case both $f(a-)$ and $f(a+)$ are defined and are equal to each other. In this case we can define $f(a)$ to be this common value, and then $f(t)$ is continuous at $t = a$.

The uniqueness theorem for the inverse Laplace transform asserts that if f and g have the same Laplace transform, then $f(a-) = g(a-)$ and $f(a+) = g(a+)$ for all a. If $f(t)$ and $g(t)$ are both continuous at a, so that $f(a-) = f(a+) = f(a)$ and $g(a-) = g(a+) = g(a)$, then it follows that $f(a) = g(a)$.

Part of the strength of the theory of the Laplace transform is its ability to deal smoothly with things like the delta function. In fact, we can form the Laplace transform of a generalized function as described in Section 17, assuming that it is of exponential type. The Laplace transform $F(s)$ determines the singular part of $f(t)$: if $F(s) = G(s)$ then $f_s(t) = g_s(t)$.

21.3. **Worrying about** $t = 0$**.** When we consider the Laplace transform of $f(t)$ in this course, we always make the assumption that $f(t) = 0$ for $t < 0$. Thus

$$f(0-) = 0.$$

What happens at $t = 0$ needs special attention, and the definition of the Laplace transform offered in Edwards and Penney (and most other ODE textbooks) is not consistent with the properties they assert.

They define

$$F(s) = \int_0^\infty e^{-st} f(t) \, dt.$$

Suppose we let $f(t) = \delta(t)$, so that $F(s)$ should be the constant function with value 1. The integrand is $\delta(t)$ again, since $e^{-st}|_{t=0} = 1$. The indefinite integral of $\delta(t)$ is the step function $u(t)$, which does not have a well-defined value at $t = 0$. Thus the value they assign to the the definite integral with lower limit 0 is ambiguous. We want the answer to be 1. This indicates that we should really define the Laplace transform of $f(t)$ as the integral

(1)
$$\boxed{F(s) = \int_{0-}^\infty e^{-st} f(t) \, dt.}$$

The integral is defined by taking the limit as the lower limit increases to zero from below. It coincides with the integral with lower limit $-\infty$ since $f(t) = 0$ for $t < 0$:

$$F(s) = \int_{-\infty}^\infty e^{-st} f(t) \, dt.$$

With this definition, $\delta(t) \rightsquigarrow 1$, as desired.

Let's now check some basic properties of the Laplace transform, using this definition.

21.4. **The** t**-derivative rule.** Integration by parts gives us

$$\int_{0-}^\infty e^{-st} f'(t) \, dt = f(0-) - (-s) \int_{0-}^\infty e^{-st} f(t) \, dt.$$

Since $f(0-) = 0$, we find that if $f(t) \rightsquigarrow F(s)$ then

(2) $$f'(t) \rightsquigarrow sF(s).$$

Here I am using the **generalized derivative** *described in Section 17.* Including the delta terms in the derivative is important even if $f(t)$ is continuous for $t > 0$ because the jump from $f(0-) = 0$ to $f(0+)$ contributes $f(0+)\delta(t)$ to $f'(t)$. Let's assume that $f(t)$ is continuous for $t > 0$. Then $f'(t) = f(0+)\delta(t) + f'_r(t)$, where $f'_r(t)$ is the regular part of the generalized derivative, that is, the ordinary derivative of the function $f(t)$. Substituting this into (2) and using $\delta(t) \rightsquigarrow 1$ and linearity gives us the familiar formula

(3) $$f'_r(t) \rightsquigarrow sF(s) - f(0+).$$

Formula (16) on p. 281 of Edwards and Penney is an awkward formulation of (2).

21.5. **The initial singularity formula.** If $f(t)$ is a generalized function with singular part at zero given by $b\delta(t)$, then

$$\lim_{s \to \infty \cdot 1} F(s) = b.$$

The notation means that we look at values of $F(s)$ for large *real* values of s.

To see this, break $f(t)$ into regular and singular parts. We have a standing assumption that the regular part $f_r(t)$ is of exponential order, and we know (from Edwards and Penney, formula (25) on p. 271 for example) that its Laplace transform dies off as $s \to \infty \cdot 1$.

Each delta function $\delta(t - a)$ in $f(t)$ contributes a term e^{-as} to $F(s)$, and as long as $a > 0$, these all decay to zero as $s \to \infty \cdot 1$ as well. Only $a = 0$ is left, and we know that $b\delta(t) \rightsquigarrow b$. This finishes the proof.

When $b = 0$—that is, when $f(t)$ is nonsingular at $t = 0$—the result is that

$$\lim_{s \to \infty \cdot 1} F(s) = 0.$$

21.6. **The initial value formula.** If $f(t)$ is a piecewise differentiable generalized function, then

$$\lim_{s \to \infty \cdot 1} sF(s) = f(0+).$$

To see this, let $f'(t)$ again denote the generalized derivative. The jump from $f(0-) = 0$ to $f(0+)$ contributes the term $f(0+)\delta(t)$ to $f'(t)$.

The initial value formula results directly from the initial singularity formula applied to $f'(t)$.

For example, if $f(t) = \cos t$ then $F(s) = \dfrac{s}{s^2 + 1}$, and

$$\lim_{s \to \infty} \frac{s^2}{s^2 + 1} = 1$$

which also happens to be $\cos(0)$. With $f(t) = \sin t$, on the other hand, $F(s) = \dfrac{1}{s^2 + 1}$, and $\lim\limits_{s \to \infty} \dfrac{s}{s^2 + 1} = 0$ in agreement with $\sin 0 = 0$.

There is also a "final value formula," which describes $\lim\limits_{t \to \infty} f(t)$ in terms of $F(s)$, but it involves the concept of a pole, so we defer it to Section 22.

21.7. **Initial conditions.** Let's return to the first order bank account model, $\dot{x} + px = q(t)$. When we come to specify initial conditions, at $t = 0$, the following two procedures are clearly equivalent: (1) Fix $x(0+) = x_0$ and proceed; and (2) Fix $x(0-) = 0$—*rest* initial conditions—but at $t = 0$ deposit a lump sum of x_0 dollars. The second can be modeled by altering the signal, replacing the rate of deposit $q(t)$ by $q(t) + x_0 \delta(t)$. Thus if you are willing to accept a generalized function for a signal, you can get always away with rest initial conditions.

Let's see how this works out in terms of the Laplace transform. In the first scenario, the t-derivative theorem gives us

$$(sX - x_0) + qX = Q.$$

In the second scenario, the fact that $\delta(t) \rightsquigarrow 1$ gives us

$$sX + qX = Q + x_0,$$

an equivalent expression.

This example points out how one can absorb certain non-rest initial conditions into the signal. The general picture is that the *top* derivative you specify as part of the initial data can be imposed using a delta function.

The mechanical model of the second degree case helps us understand this. We have an equation

$$m\ddot{x} + b\dot{x} + kx = q(t).$$

Suppose the initial *position* is $x(0) = 0$. Again, we have two alternatives: (1) Fix the initial velocity at $\dot{x}(0+) = v_0$ and proceed; or (2) Fix $\dot{x}(0-) = 0$—so, together with $x(0-) = 0$ we have rest initial

conditions—and then at $t = 0$ give the system an impulse, by adding to the signal the term $mv_0\delta(t)$. The factor of m is necessary, because we want to use this impulse to jack the velocity up to the value v_0, and the force required to do this will depend upon m.

Exercise 21.7.1. As above, check that this equivalence is consistent with the Laplace transform.

For a higher order example, consider the LTI operator $L = (D^3 + 2D^2 - 2I)$ with transfer function $W(s) = 1/(s^3 + s^2 - 2)$. In the section on the Laplace Transform in the complex plane we computed the corresponding weight function: $w(t) = e^t/5 + (e^{-t}/5)(-\cos t - 2\sin t)$ (for $t > 0$). This is a solution to the ODE $Lx = \delta(t)$ with rest initial conditions. This is equivalent (for $t > 0$) to the IVP $Lx = 0$ with initial conditions $x(0) = \dot{x}(0) = 0$ and $\ddot{x}(0) = 1$, and indeed $w(t)$ satisfies all this.

In order to capture initial conditions of lower derivatives using impulses and such in the signal, one must consider not just the delta function but also its derivatives. The desire to do this is a good motivation for extending the notion of generalized functions, but not something we will pursue in this course.