22. The pole diagram and the Laplace transform

When working with the Laplace transform, it is best to think of the variable s in $F(s)$ as ranging over the *complex* numbers. In the first section below we will discuss a way of visualizing at least some aspects of such a function—via the "pole diagram." Next we'll describe what the pole diagram of $F(s)$ tells us—and what it does not tell us—about the original function $f(t)$. In the third section we discuss the properties of the integral defining the Laplace transform, allowing s to be complex. The last section describes the Laplace transform of a periodic function of t, and its pole diagram.

22.1. **Poles and the pole diagram.** The real power of the Laplace transform is not so much as an algorithm for explicitly computing linear time-invariant system responses as in gaining insight into these responses *without* explicitly computing them. (A further feature of the Laplace transform is that it allows one to analyze systems which are not modeled by ODEs at all, by exactly the same methodology.) To achieve this insight we will have to regard the transform variable s as *complex*, and the transform function $F(s)$ as a complex-valued function of a complex variable.

A simple example is $F(s) = 1/(s-z)$, for a fixed complex number z. We can get some insight into a complex-valued function of a complex variable, such as $1/(s-z)$, by thinking about its absolute value: $|1/(s-z)| = 1/|s-z|$. This is now a *real-valued* function on the complex plane, and its graph is a surface lying over the plane, whose height over a point s is given by the value $|1/(s-z)|$. This is a tent-like surface lying over the complex plane, with elevation given by the reciprocal of the distance to z. It sweeps up to infinity like a hyperbola as s approaches z; it's as if it is being held up at $s = z$ by a tent-pole, and perhaps this is why we say that $1/(s-z)$ "has a pole at $s = z$." Generally, a function of complex numbers has a "pole" at $s = z$ when it becomes infinite there.

$F(s) = 1/(s-z)$ is an example of a **rational function**: a quotient of one polynomial by another. A product of two rational functions is again a rational function. Because you can use a common denominator, a sum of two rational functions is also a rational function. The reciprocal of any rational function except the zero function is again a rational function—exchange numerator and denominator. In these algebraic respects, the collection of rational functions behaves like the set of rational *numbers*.

Partial fractions let you write any rational function as a sum

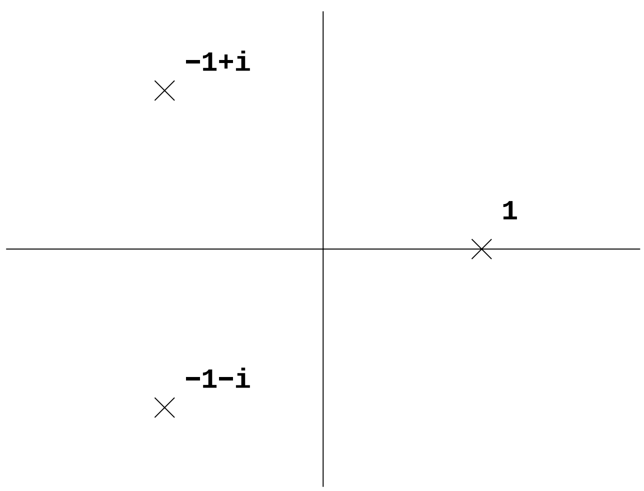
$$F(s) = p(s) + \frac{w_1}{s - z_1} + \ldots + \frac{w_n}{s - z_n} \tag{1}$$

where $p(s)$ is a polynomial and $w_1, \ldots, w_n$ and $z_1, \ldots z_n$ are complex constants. As long as the z_k's are different from each other, and the w_k's are nonzero (so the term really appears!), the poles of $F(s)$ occur exactly at the points $z_1, \ldots, z_n$ in the complex plane.

From the way the partial fraction algorithm works, the poles of a quotient of polynomials, say $N(s)/D(s)$, occur at the roots of D which don't occur also as roots of $N(s)$—or, more precisely, which occur to greater multiplicity than they do in $D(s)$.

For example, the calculation done in Section 20.5 shows that the poles of $F(s) = 1/(s^3 + s^2 - 2)$ are at $s = 1$, $s = -1 + i$, and $s = -1 - i$.

The **pole diagram** of a complex function $F(s)$ is just the complex plane with the poles of $F(s)$ marked on it. Figure 22.1 shows the pole diagram of the function $F(s) = 1/(s^3 + s^2 - 2)$.



The constant w_k appearing in 1 is the **residue** of the pole at $s = z_k$. The calculation in 20.5 shows that the residue at $s = 1$ is $1/5$, the residue at $s = -1 + 2i$ is $(-1 + 2i)/10$, and the residue at $s = -1 - 2i$ is $(-1 - 2i)/10$.

Laplace transforms are not always rational functions. For example, the exponential function occurs: $F(s) = e^{ws}$, for w a complex constant. The exponential function has *no poles*: it takes on well defined complex values for any complex input s.

We can form more elaborate complex functions by taking products—$e^{-s}/(s^3+s^2-2)$, for example. The numerator doesn't contribute any poles. Nor does it kill any poles—it is never zero, so it doesn't cancel any of the roots of the denominator. The pole diagram of this function is the same as the pole diagram of $1/(s^3+s^2-2)$.

A general complex function of the type that occurs as a Laplace transform (the mathematical term is *meromorphic*) does not have a partial fraction decomposition, so we can't use (1) to locate the poles. Poles occur where the value of the function blows up. This can be expressed as follows. Define the **residue** of $F(s)$ at $s=z$ as

$$\text{res}_{s=z}F(s) = \lim_{s\to z}(s-z)F(s). \tag{2}$$

If $F(s)$ takes on a complex value at $s=z$, or if at least $\lim_{s\to z} F(s)$ does so, then the residue at $s=z$ is zero. In this case $s=z$ is not a pole of $F(s)$. Poles occur at the places where the residue is nonzero.

A complex function is by no means completely specified by its pole diagram. Nevertheless, the pole diagram of $F(s)$ carries a lot of information about $F(s)$, and if $F(s)$ is the Laplace transform of $f(t)$, it tells you a lot of information of a specific type about $f(t)$.

22.2. **The pole diagram of the Laplace transform.**

Summary: The pole diagram of $F(s)$ tells us about *long-term behavior* of $f(t)$. It tells us *nothing* about the near-term behavior.

This is best seen by examples.

Suppose we have just one pole, at $s=1$. Among the functions with this pole diagram we have:

$$F(s) = \frac{c}{s-1}\,, \quad G(s) = \frac{ce^{-as}}{s-1}\,, \quad H(s) = \frac{c}{s-1} + b\frac{1-e^{-as}}{s}$$

where $c \neq 0$. (Note that $1-e^{-as}$ becomes zero when $s=0$, canceling the zero in the denominator of the second term in $H(s)$.) To be Laplace transforms of real functions we must also assume them all to be real, and $a \geq 0$. Then these are the Laplace transforms of

$$f(s) = ce^t\,, \quad g(t) = \begin{cases} ce^{t-a} & \text{for } t > a, \\ 0 & \text{for } t < a \end{cases}\,, \quad h(t) = \begin{cases} ce^t & \text{for } t > a, \\ ce^t + b & \text{for } t < a \end{cases}$$

All these functions grow like a multiple of e^t when t is large. You can even say which multiple: it is given by the residue at $s=1$. (Note that $g(t) = (ce^{-a})e^t$, and the residue of $G(s)$ at $s=1$ is ce^{-a}.) But their behavior when $t < a$ is all over the map. In fact, the function can be *anything* for $t < a$, for *any* fixed a; as long as it settles down to something close to ce^t for t large, its Laplace transform will have just one pole, at $s = 1$, with residue c.

Now suppose we have two poles, say at $s = a + bi$ and $s = a - bi$. Two functions with this pole diagram are

$$F(s) = \frac{c(s-a)}{(s-a)^2 + b^2}\,, \quad G(s) = \frac{cb}{(s-a)^2 + b^2}.$$

and we can modify these as above to find others. These are the Laplace transform of

$$f(t) = ce^{at}\cos(bt)\,, \quad g(t) = ce^{at}\sin(bt).$$

This reveals that it is the *real part* of the pole that determines the long term *growth in maximum magnitude*. The *imaginary part* of the pole determines the *circular frequency of oscillation* for large t. We can't pick out the phase—it can oscillate like a sine or a cosine, or exhibit any other phase shift. And we can't promise that it will be exactly sinusoidal times exponential, but it will resemble this. And again, the pole diagram of $F(s)$ says *nothing* about $f(t)$ for small t.

Now let's combine several of these, to get a function with several poles. Suppose $F(s)$ has poles at $s = 1$, $s = -1 + i$, and $s = -1 - i$, for example. We should expect that $f(t)$ has a term which grows like e^t (from the pole at $s = 1$), and another term which behaves like $e^{-t}\cos t$ (up to constants and phase shifts). When t is large, the damped oscillation becomes hard to detect as the other term grows exponentially.

We learn that the *rightmost poles dominate*—the ones with *largest real part* have the dominant influence on the long-term behavior of $f(t)$.

The most important consequence relates to the question of *stability*:

> If all the poles of $F(s)$ have *negative real part* then $f(t)$ decays exponentially to zero as $t \to \infty$.

If some pole has positive real part, then $|f(t)|$ becomes arbitrarily large for large t.

In summary:

The position of the rightmost poles of $F(s)$ determine the general behavior of $f(t)$ for large time. If the rightmost pole is at $a+bi$, then $f(t)$ will behave roughly like a multiple of $e^{(a+bi)t}$ for t large; that is, it will grow (or decay) approximately like the function e^{at}, and oscillate approximately like $\cos(bt)$.

There is a further subtlety here: the *order* of the pole contributes a *polynomial factor* to the rate of growth. This can be seen in the example $t^{n-1} \rightsquigarrow n!/s^n$.

Here's the general picture, with all the detail in place. To determine the large-time behavior of $f(t)$, look at the rightmost poles of $F(s)$; say they occur along the line $\mathrm{Re}\,(s) = a$. Among them, take those of maximal order, say n. Then there is a constant C for which $|f(t)| < Ct^n e^{at}$ for all large t. The real number a is minimal with this property, and given a the integer n is minimal with this property. Finally, if there's only one such pole, say at $s = a + bi$, then $f(t)$ oscillates with approximate circular frequency b.

Comment on reality. We have happily taken the Laplace transform of complex valued functions of t: $e^{it} \rightsquigarrow 1/(s-i)$, for example. If $f(t)$ is real, $F(s)$ enjoys a symmetry with respect to complex conjugation:

(3) $\boxed{\text{If } f(t) \text{ is real-valued then } F(\overline{s}) = \overline{F(s)}.}$

The pole diagram of a function $F(s)$ with this property is symmetric about the real axis: non-real poles occur in complex conjugate pairs. Thus:

The pole diagram of the Laplace transform of a real function is symmetric across the real axis.

22.3. **The Laplace transform integral.** In the integral defining the Laplace transform, we really should let s be complex. We are thus integrating a complex-valued function of a real parameter t, $e^{-st}f(t)$, and this is done by integrating the real and imaginary parts separately. It is an improper integral, computed as the limit of $\displaystyle\int_0^T e^{-st}f(t)\,dt$ as $T \to \infty$. [Actually, we will see in Section 21 that it's better to think of the lower limit as "improper" as well, in the sense that we form the integral with lower limit $a < 0$ and then let $a \uparrow 0$.] The textbook assumption that $f(t)$ is of "exponential order" is designed so that if s has large enough real part, the term e^{-st} will be so small (at least for large t) that the product $e^{-st}f(t)$ has an integral which stays bounded as $T \to \infty$. In terms of the pole diagram, we may say that the integral converges when the real part of s is bigger than the real part of any pole in the resulting transform function $F(s)$. The exponential order assumption is designed to guarantee that we won't get poles with arbitrarily large real part.

The region to the right of the rightmost pole is called the **region of convergence**. Engineers abbreviate this and call it the "ROC."

Once the integral has been computed, the expression in terms of s will have meaning for all complex numbers s (though it may have a pole at some).

For example, let's consider the time-function $f(t) = 1$, $t > 0$. Then:

$$F(s) = \int_0^\infty e^{-st}dt = \lim_{T\to\infty} \frac{e^{-st}}{-s}\bigg|_0^T = \frac{1}{-s}\left(\lim_{T\to\infty} e^{-sT} - 1\right).$$

Since $\left|e^{-sT}\right| = e^{-aT}$ if $s = a + bi$, the limit is 0 if $a > 0$ and doesn't exist if $a < 0$. If $a = 0$, $e^{-sT} = \cos(bT) - i\sin(bT)$, which does not have a limit as $T \to 0$ unless $b = 0$ (which case is not relevant to us since we certainly must have $s \neq 0$). Thus the improper integral converges exactly when $\mathrm{Re}\,(s) > 0$, and gives $F(s) = 1/s$. Despite the fact that the integral definitely diverges for $\mathrm{Re}\,(s) \leq 0$, the expression $1/s$ makes sense for all $s \in \mathbb{C}$ (except for $s = 0$), and it's better to think of the function $F(s)$ as defined everywhere in this way. This process is called **analytic continuation**.

22.4. **Final value formula.** Suppose that $f(t)$ has only finitely many points of discontinuity and that its singular part has only finitely many delta functions in it. If all poles of $F(s)$ are to the left of the imaginary axis, then

$$\lim_{t\to\infty} f(t) = 0.$$

If there is a simple pole at $s = 0$ and all other poles are to the left of the imaginary axis, then

$$\lim_{t\to\infty} f(t) = \mathrm{res}_{s=0} F(s).$$

In other situations, $\lim_{t\to\infty} f(t)$ doesn't exist; either $f(t)$ oscillates without decaying, or $|f(t)|$ grows without bound.

We will not attempt to justify this in detail, but notice that c/s has a pole at $s = 0$ with residue c (see Section 22, (2)) and its inverse Laplace transform is the constant function c, which certainly has "final value" c.

The final value formula implies a description of the behavior of $f(t)$ as $t \to \infty$ for a much broader class of functions $f(t)$. Suppose that the rightmost pole of $F(s)$ has real part strictly less than a. By the s-shift rule, the Laplace transform of $e^{-at}f(t)$ is $F(s+a)$. The pole diagram of $F(s+a)$ is the same as the pole diagram of $F(s)$ but shifted to the left by a units in the complex plane. Thus all its poles are to the left of the imaginary axis, and so $\lim_{t\to\infty} e^{-at}f(t) = 0$. This says that $f(t)$ grows more slowly than e^{at}, or (if $a < 0$) decays faster than e^{at}.

Similarly, if $F(s)$ has one pole whose real part is larger than those of all its other poles, and that pole is real—say it's a—then

$$\lim_{t\to\infty} e^{at}f(t) = \mathrm{res}_{s=a}F(s).$$

If the pole at $s = a$ is simple, so the residue is finite, then this says that $f(t)$ grows or decays like a constant multiple of the exponential function e^{-at}.

If on the other hand there is a rightmost pole which is not real (so if $f(t)$ is to be real there is at least a complex conjugate pair of rightmost poles) then the function oscillates. If there is one conjugate pair of rightmost poles, $a \pm i\omega$, then $f(t)$ oscillates with an overall approximate circular frequency of ω, while growing or decaying approximately as fast as e^{-at}. This can be seen in the examples

$$e^{at}\cos(\omega t) \rightsquigarrow \frac{s-a}{(s-a)^2+\omega^2}\,, \quad e^{at}\sin(\omega t) \rightsquigarrow \frac{\omega}{(s-a)^2+\omega^2}.$$

In these cases the poles of $F(s)$ are at $a \pm i\omega$.

This is an important principle: the general behavior of $f(t)$ as $t \to \infty$ is controlled by the pole diagram of its Laplace transform $F(s)$; in fact, its main features are controlled by the rightmost poles in the pole diagram.

22.5. **Laplace transform and Fourier series.** We now have two ways to understand the harmonic components of a periodic function $f(t)$. First, we can form the Laplace transform $F(s)$ of $f(t)$ (regarded as defined only for $t > 0$). Since $f(t)$ is periodic, the poles of $F(s)$ lie entirely along the imaginary axis, and the locations of these poles reveal periodic constituents in $f(t)$. On the other hand, $f(t)$ has a Fourier series, which explicitly expresses it as a sum of sinusoidal components. What is the relation between these two perspectives?

For example, the Fourier series of the squarewave $\mathrm{sq}(t)$ of period 2π, with value 1 for $0 < t < \pi$ and -1 for $-\pi < t < 0$, is

$$\mathrm{sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right).$$

The circular frequencies of the Fourier components of $\mathrm{sq}(t)$ are $1, 3, 5, \ldots$. In the notes on Laplace transform we studied the Laplace transform of periodic functions (restricted to the interval $(0, \infty)$), and found among other things that the poles of the Laplace transform of $\mathrm{sq}(t)$ (regarded now as defined for $t > 0$) occur at $\pm i, \pm 3i, \pm 5i, \ldots$. These poles reveal the presence in the squarewave of oscillations with circular frequencies $1, 3, 5, \ldots$, but *none* with even circular frequencies, a somewhat surprising circumstance confirmed by the Fourier series.

It is easy to see the connection in general, especially if we use the complex form of the Fourier series,

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}.$$

Simply apply the Laplace transform to this expression, using $e^{int} \rightsquigarrow \dfrac{1}{s - in}$:

$$F(s) = \sum_{n=-\infty}^{\infty} \frac{c_n}{s - in}$$

The only possible poles are at the complex numbers $s = in$, and the residue at in is c_n.

The same conclusion holds if $f(t)$ has period $2a$:

> If $f(t)$ is periodic of period $2a$, the poles of $F(s)$ occur only at points of the form $k\pi i/a$ for k an integer, and the residues at these poles are precisely the complex Fourier coefficients c_n of $f(t)$.

In our example of the squarewave, $a_n = 0$ for all n, $b_n = 0$ for n even, and $b_n = 4/n\pi$ for n odd. The predicted residues are thus 0 at $\pm ni$ for n even (so no pole there), and $2/n\pi i$ at $s = ni$ for n odd, in agreement with our earlier calculation.