23. The Laplace transform and more general systems

This section gives a hint of how flexible a device the Laplace transform is in engineering applications.

23.1. **Zeros of the Laplace transform: stillness in motion.**

The mathematical theory of functions of a complex variable shows that the *zeros* of $F(s)$—the values r of s for which $F(r) = 0$—are just as important to our understanding of it as are the poles. This symmetry is reflected in engineering as well; the location of the zeros of the transfer function has just as much significance as the location of the poles. Instead of recording resonance, they reflect stillness.

Example 23.1.1. We envision the following double spring system: there is an object with mass m_1 suspended by a spring with spring constant k_1. A second object with mass m_2 is suspended from this first object by a second spring with constant k_2. The system is driven by motion of the top of the top spring according to a function $f(t)$. Pick coordinates so that x_1 is the position of the first object and x_2 is the position of the second, both increasing in the downward direction, and both zero at rest. Direct the position function $f(t)$ downwards also. Figure 13 shows this system.

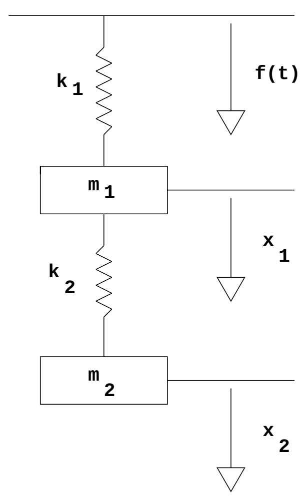

Figure 13. Two spring system

The equations of motion are

$$\left\{ \begin{array}{rcl} m_1\ddot{x}_1 & = & k_1(f(t) - x_1) - k_2(x_1 - x_2) \\ m_2\ddot{x}_2 & = & k_2(x_1 - x_2) \end{array} \right. \tag{1}$$

This is a *system of second order equations*, and as you can imagine mechanical engineering is full of similar systems.

Suppose that our main interest is in x_1. Let's take Laplace transforms, and assume rest initial conditions.

$$\begin{cases} (m_1 s^2 + (k_1 + k_2))X_1 &= k_2 X_2 + k_1 F \\ (m_2 s^2 + k_2)X_2 &= k_2 X_1. \end{cases}$$

Use the second equation to express X_2 in terms of X_1, and substitute this value into the first equation. Then solve for X_1 to get:

$$X_1(s) = \frac{m_2 s^2 + k_2}{(m_1 s^2 + (k_1 + k_2))(m_2 s^2 + k_2) - k_2^2} \cdot k_1 F(s).$$

The "transfer function" $W(s)$ is then the ratio of the LT of the system response, X_1, and the LT of the input signal, F:

$$W(s) = \frac{k_1(m_2 s^2 + k_2)}{(m_1 s^2 + (k_1 + k_2))(m_2 s^2 + k_2) - k_2^2}.$$

It is still the case that $W(r)$ is the multiple of e^{rt} which occurs as x_1 in a solution to the equations (1) when we take $f(t) = e^{rt}$. Thus the zeros of $W(s)$ at $s = \pm i\sqrt{k_2/m_2}$—the values of s for which $W(s) = 0$—reflect a "neutralizing" circular frequency of $\omega = \sqrt{k_2/m_2}$. If $f(t)$ is sinusoidal of this circular frequency then $x_1 = 0$ is a solution. The suspended weight oscillates with (k_1/k_2) times the amplitude of $f(t)$ and reversed in phase (independent of the masses!), and exactly cancels the impressed force. Check it out!

Example 23.1.2. To see another way in which a transfer function can have a nonconstant numerator, let's model a shock absorber. A platform is caused to move vertically according to a displacement function $y(t)$. A mass m is supported by a spring (with spring constant k) and a dashpot (with damping constant b), arranged in parallel so to speak. Use a variable x to measure the the elevation of the mass, and arrange that when $x = y$ the spring is relaxed. Ignore gravity. The system is diagrammed in Figure 14.

The equation of motion is

$$m\ddot{x} = k(y - x) + b(\dot{y} - \dot{x})$$

If we put the "system" on the left we have

$$m\ddot{x} + b\dot{x} + kx = b\dot{y} + ky. \tag{2}$$

Now, in 18.03 we would tend to consider the "input signal" to be the right hand side here. But it is more meaningful to consider the

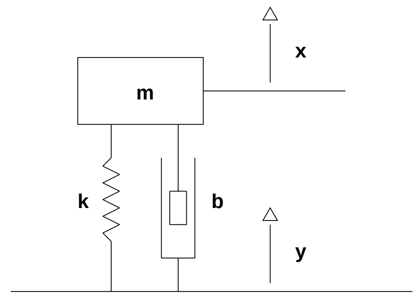

FIGURE 14. A car suspension system

input signal to be simply y. After all, if the platform is actually the chassis of a car, bumping over a Cambridge street, then it's y that's making things happen. We would like to get from knowledge of y to the solution of this equation (with rest initial conditions, say). If we apply the Laplace transform to both sides we find $p(s)X = (bs + k)Y$ where $p(s) = ms^2 + bs + k$ is the characteristic polynomial of the system. We can solve:

$$X = WY, \qquad W(s) = \frac{bs + k}{ms^2 + bs + k}.$$

Remember the significance of the transfer function: $W(r)$ is the multiple of the signal e^{rt} which occurs as a system response to the input signal e^{rt}. Since $W(-k/b) = 0$, the system response to $y = e^{-kt/b}$ is $x = 0$, constant. In fact, the right hand side of (2) is then zero, so the solutions are the solutions of the homogeneous equation $m\ddot{x} + b\dot{x} + kx = 0$.

23.2. **General LTI systems.** The weight function $w(t)$, or its Laplace transform, the transfer function $W(s)$, completely determine the system. The transfer function of an ODE has a very restricted form—it is the reciprocal of a polynomial; but the mechanism for determining the system response makes sense for much more general complex functions $W(t)$, and, correspondingly, much more general "weight functions" $w(t)$: given a very general function $w(t)$, we can define an LTI system by declaring that a signal $f(t)$ results in a system response (with null initial condition, though in fact nontrivial initial conditions can be handled too, by absorbing them into the signal using delta functions) given by the convolution $f(t) * w(t)$. The apparatus of the Laplace transform helps us, too, since we can compute this system response as the inverse Laplace transform of $F(s)W(s)$. This mechanism allows us to represent the *system*, the *signal*, and the *system response*, all three, using *functions* (of t, or of s). Differential operators have vanished from the scene. This flexibility results in a tool of tremendous power.