3. Solutions of first order linear ODEs

3.1. Homogeneous and inhomogeneous; superposition. A first order linear equation is **homogeneous** if the right hand side is zero:

$$\dot{x} + p(t)x = 0\,. \tag{1}$$

Homogeneous linear equations are separable, and so the solution can be expressed in terms of an integral. The general solution is

$$x = \pm\, e^{-\int p(t)dt} \qquad \text{or} \quad x = 0\,. \tag{2}$$

Question: Where's the constant of integration here? Answer: The indefinite integral is only defined up to adding a constant, which becomes a positive factor when it is exponentiated.

We also have the option of replacing the indefinite integral with a definite integral. The lower bound will be some value of t at which the ODE is defined, say a, while the upper limit should be t, in order to define a function of t. This means that I have to use a different symbol for the variable inside the integral—say τ, the Greek letter "tau." The general solution can then be written as

$$x = c\, e^{-\int_a^t p(\tau)d\tau}\,, \quad c \in \mathbb{R}\,. \tag{3}$$

This expression for the general solution to (1) will often prove useful, even when it can't be integrated in elementary functions. Note that the constant of integration is also an initial value: $c = x(a)$.

I am not requiring $p(t)$ to be constant here. If it is, then we can evaluate the integral and find the familiar solution $x = ce^{-pt}$.

These formulas tell us something important about a function $x = x(t)$ which satisfies (1): either $x(t) = 0$ for *all* t, or $x(t) \neq 0$ for *all* t: either x is the zero function, or it's never zero. This is a consequence of the fact that the exponential function never takes on the value zero.

Even without solving it, we can observe an important feature of the solutions of (1):

If x_h is a solution, so is cx_h for any constant c.

The subscripted h is for "homogeneous." This can be verified directly by assuming that x_h is a solution and then checking that cx_h is too. Conversely, if x_h is *any* nonzero solution, then the *general* solution is cx_h: *every* solution is a multiple of x_h. This is because of the uniqueness theorem for solutions: for any choice of initial value $x(a)$, I can find c

so that $cx_h(a) = x(a)$ (namely, $c = x(a)/x_h(a)$), and so by uniqueness $x = cx_h$ for this value of c.

Now suppose the input signal is nonzero, so our equation is

$$\dot{x} + p(t)x = q(t)\,. \tag{4}$$

Suppose that in one way or another we have found a solution x_p to (4). *Any* single solution will do. We will call it a "particular" solution. Keeping the notation x_h for a nonzero solution to the corresponding *homogeneous* equation (1), we can calculate that $x_p + cx_h$ is again a solution to (4).

Exercise 3.1.1. Verify this.

In fact,

$$\boxed{\text{The general solution to (4) is } x_p + cx_h} \tag{5}$$

since any initial condition can be achieved by judicious choice of c. This formula shows how the constant of integration, c, occurs in the general solution of a *linear* equation. It tends to show up in a more complicated way if the equation is nonlinear.

I want to emphasize that despite being called "particular," the solution x_p can be *any* solution of (4); it need not be *special* in any way for it to serve in (5).

There's a slight generalization: suppose x_1 is a solution to

$$\dot{x} + p(t)x = q_1(t)$$

and x_2 is a solution to

$$\dot{x} + p(t)x = q_2(t)$$

—same coefficient $p(t)$, so the same system, but two different input signals. Then (for any constants c_1, c_2) $c_1x_1 + c_2x_2$ is a solution to

$$\dot{x} + p(t)x = c_1q_1(t) + c_2q_2(t)\,.$$

In our banking example, if we have two bank accounts with the same interest rate, and contribute to them separately, the sum of the accounts will be the same as if we combined them into one account and contributed the sum to the combined account. This is the **principle of superposition**.

The principle of superposition lets us break up the input signal into bitesized pieces, solve the corresponding equations, and add the solutions back together to get a solution to the original equation.

3.2. **Variation of parameters.** Now we try to solve the general first order linear equation,

$$\dot{x} + p(t)x = q(t)\,. \tag{6}$$

As we presented it above, the procedure for solving this breaks into two parts. We first find a nonzero solution, say x_h, of the **associated homogeneous equation**

$$\dot{x} + p(t)x = 0 \tag{7}$$

—that is, (6) with the right hand side replaced by zero. *Any* nonzero solution will do, and since (7) is separable, finding one is a matter of integration. The general solution to (7) is then cx_h for a constant c. The constant c "parametrizes" the solutions to (7).

The second step is to somehow find some single solution to (6) itself. We have not addressed this problem yet. One idea is to hope for a solution of the form vx_h, where v now is *not* a constant (which would just give a solution to the *homogeneous* equation), but rather some function of t, which we will write as $v(t)$ or just v.

So let's make the substitution $x = vx_h$ and study the consequences. When we make this substitution in (6) and use the product rule we find

$$\dot{v}x_h + v\dot{x}_h + pvx_h = q\,.$$

The second and third terms sum to zero, since x_h is a solution to (7), so we are left with a differential equation for v:

$$\dot{v} = x_h^{-1}q\,. \tag{8}$$

This can be solved by direct integration once again. Write v_p for a particular solution to (8). A particular solution to our original equation (6) is then given by $x_p = v_p x_h$.

By superposition, the general solution is $x = x_p + cx_h$. You can also see this by realizing that the general solution to (8) is $v = v_p + c$, so the general solution x is $vx_h = x_p + cx_h$.

Many people like to remember this in the following form: the general solution to (6) is

$$\boxed{x = x_h \int x_h^{-1} q\,dt} \tag{9}$$

since the general solution to (8) is $v = \displaystyle\int x_h^{-1} q\,dt$. Others just make the substitution $x = vx_h$ and do the calculation.

Example. The inhomogeneous first order linear ODE we wish to solve is

$$\dot{x} + tx = (1+t)e^t \, .$$

The associated homogeneous equation is

$$\dot{x} + tx = 0,$$

which is separable and easily leads to the nonzero solution $x_h = e^{-t^2/2}$. So we'll try for a solution of the original equation of the form $x = ve^{-t^2/2}$. Substituting this into the equation and using the product rule gives us

$$\dot{v}e^{-t^2/2} - vte^{-t^2/2} + vte^{-t^2/2} = (1+t)e^t \, .$$

The second and third terms cancel, as expected, leaving us with $\dot{v} = (1+t)e^{t+t^2/2}$. Luckily, the derivative of the exponent here occurs as a factor, so this is easy to integrate: $v_p = e^{t+t^2/2}$ (plus a constant, which we might as well take to be zero since we are interested only in finding one solution). Thus a particular solution to the original equation is $x_p = v_p x_h = e^t$. It's easy to check that this is indeed a solution! By (5) the general solution is $x = e^t + ce^{-t^2/2}$.

This method is called "variation of parameter." The "parameter" is the constant c in the expression cx_h for the general solution of the associated homogeneous equation. It is allowed to vary with time in an effort to come up with a solution of the given *inhomogeneous* equation. The method of variation of parameter is equivalent to the method of integrating factors described in Edwards and Penney; in fact x_h^{-1} is an integrating factor for (6). Either way, we have broken the original problem into two problems each of which can be solved by direct integration.

3.3. **Continuation of solutions.** There is an important theoretical outcome of the method of Variation of Parameters. To see the point, consider first the *nonlinear* ODE $\dot{x} = x^2$. This is separable, with general solution $x = 1/(c-t)$. There is also a "missing solution" $x = 0$ (which corresponds to $c = \infty$).

As we pointed out in Section 1, the statement that $x = 1/(c-t)$ is a solution is somewhat imprecise. This equation actually defines *two* solutions: one defined for $t < c$, and another defined for $t > c$. These are *different* solutions. One becomes asymptotic to $t = c$ as $t \uparrow c$; the other becomes asymptotic to $t = c$ as $t \downarrow c$. Neither of these solutions can be extended to a solution defined at $t = c$; both solutions "blow up" at $t = c$. This pathological behavior occurs *despite* the fact that the ODE itself doesn't exhibit any special pathology at $t = c$ for any value of c.

With the exception of the constant solution, *no solution can be defined for all time*, despite the fact that the equation is perfectly well defined for all time.

Another thing that may happen to solutions of nonlinear equations is illustrated by the equation $\dot{x} = -x/y$. This is separable, and in implicit form the general solution is $x^2 + y^2 = c^2$, $c > 0$: circles centered at the origin. To get a *function* as a solution, one must restrict to the upper half plane or to the lower half plane: $y = \pm\sqrt{c^2 - x^2}$. In any case, these solutions can't be extended to all time, once again, but now for a different reason: they come up to a point at which the tangent line becomes vertical (at $x = \pm c$), and the solution function doesn't extend past that point.

The situation for *linear* equations is quite different. The fact that continuous functions are integrable (from calculus) shows that if $f(t)$ is defined and continuous on an interval, then all solutions to $\dot{x} = f(t)$ extend over the same interval. Because the solution to (6) is achieved by two direct integrations, we obtain the following result, which stands in contrast to the situation typical of nonlinear equations.

Theorem: If p and q are defined (and reasonably well-behaved) for all t between a and b, then any solution to $\dot{x} + p(t)x = q(t)$ defined somewhere between a and b extends to a solution defined on the entire interval from a to b.

3.4. **Final comments on the bank account model.** Let us solve (1) in the special case in which I and q are both constant. In this case the equation

$$\dot{x} - Ix = q$$

is separable; we do not need to use the method of variation of parameters or integrating factors. Separating,

$$\frac{dx}{x + q/I} = I\,dt$$

so integrating and exponentiating,

$$x = -q/I + ce^{It}\,, \quad c \in \mathbb{R}\,.$$

Let's look at this formula for a moment. There is a constant solution, namely $x = -q/I$. I call this the *credit card solution*. I owe the bank q/I dollars. They "give" me interest, at the rate of I times the value of the bank account. Since that value is negative, what they are doing is charging me: I am using the bank account as a loan, and my "contributions" amount to interest payments on the loan, and exactly balance the interest charges. The bank balance never changes. This steady state solution has large magnitude if my rate of payments is large, or if the interest is small.

If $c < 0$, I owe the bank more than can be balanced by my payments, and my debt increases exponentially. Let's not dwell on this unfortunate scenario, but pass quickly to the case $c > 0$, when some of my payments are used to pay off the principal, and ultimately to add to a positive bank balance. That balance then proceeds to grow approximately exponentially.

In terms of the initial condition $x(0) = x_0$, the solution is

$$x = -q/I + (x_0 + q/I)e^{It}\,.$$

In calling this the credit card solution, I am assuming that $q > 0$. If $q < 0$, then the constant solution $x = -q/I$ is positive. What does this signify?