

GS. Graphing ODE Systems

1. The phase plane.

Up to now we have handled systems analytically, concentrating on a procedure for solving linear systems with constant coefficients. In this chapter, we consider methods for sketching graphs of the solutions. The emphasis is on the word *sketching*. Computers do the work of drawing reasonably accurate graphs. Here we want to see how to get quick qualitative information about the graph, without having to actually calculate points on it.

First some terminology. The sort of system for which we will be trying to sketch the solutions is one which can be written in the form

$$(1) \quad \begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned}$$

Such a system is called **autonomous**, meaning the independent variable (which we understand to be t) does not appear explicitly on the right, though of course it lurks in the derivatives on the left. The system (1) is a *first-order autonomous system*; it is in *standard form* — the derivatives on the left, the functions on the right.

A **solution** of such a system has the form (we write it two ways):

$$(2) \quad \mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad \begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned}$$

It is a vector function of t , whose components satisfy the system (1) when they are substituted in for x and y . In general, you learned in 18.02 and physics that such a vector function describes a motion in the xy -plane; the equations in (2) tell how the point (x, y) moves in the xy -plane as the time t varies. The moving point traces out a curve called the **trajectory** of the solution (2). The xy -plane itself is called the **phase plane** for the system (1), when used in this way to picture the trajectories of its solutions.

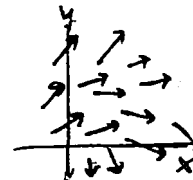
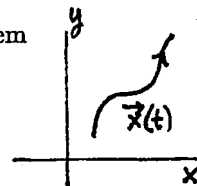
That is how we can picture the solutions (2) to the system; how can we picture the system (1) itself? We can think of the derivative of a solution

$$(3) \quad \mathbf{x}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

as representing the **velocity vector** of the point (x, y) as it moves according to (2). From this viewpoint, we can interpret geometrically the system (1) as prescribing for each point (x_0, y_0) in the xy -plane a velocity vector having its tail at (x_0, y_0) :

$$(4) \quad \mathbf{x}' = \begin{pmatrix} f(x_0, y_0) \\ g(x_0, y_0) \end{pmatrix} = f(x_0, y_0)\mathbf{i} + g(x_0, y_0)\mathbf{j}.$$

The system (1) is thus represented geometrically as a vector field, the **velocity field**. A solution (2) of the system is a point moving in the xy -plane so that at each point of its trajectory, it has the velocity prescribed by the field. The trajectory itself will be a curve which at each point has the direction of the velocity vector at that point. (An arrowhead is put on the trajectory to show the sense in which t is increasing.)



Sketching trajectories ought to remind you a lot of the work you did drawing integral curves for direction fields to first-order ODE's. What is the relation?

2. First-order autonomous ODE systems and first-order ODE's.

We can eliminate t from the first-order system (1) by dividing one equation by the other. Since by the chain rule

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt},$$

we get after the division a single first-order ODE in x and y :

$$(5) \quad \begin{array}{l} x' = f(x, y) \\ y' = g(x, y) \end{array} \quad \longrightarrow \quad \frac{dy}{dx} = \frac{g(x, y)}{f(x, y)}.$$

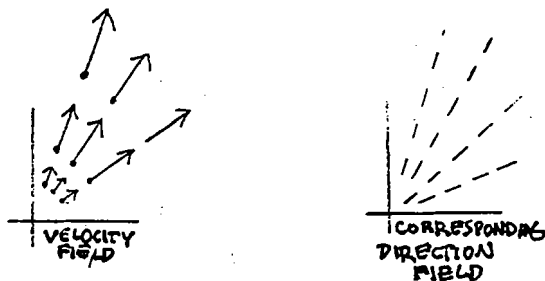
If the first order equation on the right is solvable, this is an important way of getting information about the solutions to the system on the left. Indeed, in the older literature, little distinction was made between the system and the single equation — “solving” meant to solve either one.

There is however a difference between them: the system involves time, whereas the single ODE does not. Consider how their respective solutions are related:

$$(6) \quad \begin{array}{l} x = x(t) \\ y = y(t) \end{array} \quad \longrightarrow \quad F(x, y) = 0,$$

where the equation on the right is the result of eliminating t from the pair of equations on the left. Geometrically, $F(x, y) = 0$ is the equation for the *trajectory* of the solution $\mathbf{x}(t)$ on the left. The trajectory in other words is the path traced out by the moving point $(x(t), y(t))$; it doesn't contain any record of how fast the point was moving; it is only the track (or *trace*, as one sometimes says) of its motion.

In the same way, we have the difference between the *velocity field*, which represents the left side of (5), and the *direction field*, which represents the right side. The velocity vectors have magnitude and sense, whereas the line segments that make up the direction field only have slope. The passage from the left side of (5) to the right side is represented geometrically by changing each of the velocity vectors to a line segment of standard length. Even the arrowhead is dropped, since it represents the direction of increasing time, and time has been eliminated; only the slope of the vector is retained.



In considering how to sketch trajectories of the system (1), the first thing to consider are the *critical points* (sometimes called *stationary points*).

Definition 2.1 A point (x_0, y_0) is a **critical point** of the system (1) if

$$(7a) \quad f(x_0, y_0) = 0, \quad \text{and} \quad g(x_0, y_0) = 0$$

or equivalently, if

$$(7b) \quad x = x_0, \quad y = y_0 \quad \text{is a solution to (1).}$$

The equations of the system (1) show that (7a) and (7b) are equivalent — either implies the other.

If we adopt the geometric viewpoint, thinking of the system as represented by a velocity vector field, then a critical point is one where the velocity vector is zero. Such a point is a trajectory all by itself, since by not moving it satisfies the equations (1) of the system (this explains the alternative designation “stationary point”).

The critical points represent the simplest possible solutions to (1), so you begin by finding them; by (7a), this is done by solving the pair of simultaneous equations

$$(8) \quad \begin{aligned} f(x, y) &= 0 \\ g(x, y) &= 0 \end{aligned}$$

Next, you can try the strategy indicated in (5) of passing to the associated first-order ODE and trying to solve that and sketch the solutions; or you can try to locate some sketchable solutions to (1) and draw them in. None of this is likely to work if the functions $f(x, y)$ and $g(x, y)$ on the right side of the system (1) aren't simple, but for linear equations with constant coefficients, both procedures are helpful as we shall see in the next section.

A principle that was important in sketching integral curves for direction fields applies also to sketching trajectories of the system (1): assuming the functions $f(x, y)$ and $g(x, y)$ are smooth (i.e., have continuous partial derivatives), we have the

$$(9) \quad \textbf{Sketching principle.} \quad \textit{Two trajectories of (1) cannot intersect.}$$

The sketching principle is a consequence of the existence and uniqueness theorem for systems of the form (1), which implies that in a region where the partial derivatives of f and g are continuous, through any point passes one and only one trajectory.

3. Sketching some basic linear systems. We use the above ideas to sketch a few of the simplest linear systems, so as to get an idea of the various possibilities for their trajectories, and introduce the terminology used to describe the resulting geometric pictures.

Example 3.1 Let's consider the linear system on the left below. Its characteristic equation is $\lambda^2 - 1 = 0$, so the eigenvalues are ± 1 , and it is easy to see its general solution is the one on the right below:

$$(10) \quad \begin{cases} x' = y \\ y' = x \end{cases} ; \quad \mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} .$$

$$(10) \quad \begin{cases} x' = y \\ y' = x \end{cases} ; \quad \mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} .$$

By (8), the only critical point of the system is $(0, 0)$. We try the strategy in (5); this converts the system to the first-order ODE below, whose general solution (on the right) is found by separation of variables:

$$(11) \quad \frac{dy}{dx} = \frac{x}{y} ; \quad \text{general solution: } y^2 - x^2 = c .$$

Plotted, these are the family of hyperbolas having the diagonal lines $y = \pm x$ as asymptotes; in addition there are the two lines themselves, corresponding to $c = 0$; see fig. 1.

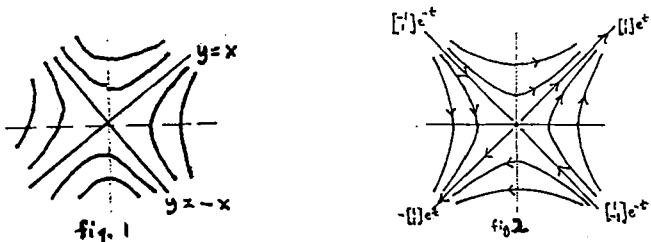
This shows us what the trajectories of (7) must look like, though it does not tell us what the direction of motion is. A further difficulty is that the two lines cross at the origin, which seems to violate the sketching principle (9) above.

We turn therefore to another strategy: plotting simple trajectories that we know. Looking at the general solution in (10), we see that by giving one of the c 's the value 0 and the other one the value 1 or -1 , we get four easy solutions:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t, \quad -\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t, \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}, \quad -\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} .$$

These four solutions give four trajectories which are easy to plot. Consider the first, for example. When $t = 0$, the point is at $(1, 1)$. As t increases, the point moves outward along the line $y = x$; as t decreases through negative values, the point moves inwards along the line, toward $(0, 0)$. Since t is always understood to be increasing on the trajectory, the whole trajectory consists of the ray $y = x$ in the first quadrant, excluding the origin (which is not reached in finite negative time), the direction of motion being outward.

A similar analysis can be made for the other three solutions; see fig. 2 below.



As you can see, each of the four solutions has as its trajectory one of the four rays, with the indicated direction of motion, outward or inward according to whether the exponential factor increases or decreases as t increases. There is even a fifth trajectory: the origin itself, which is a stationary point, i.e., a solution all by itself. So the paradox of the intersecting diagonal trajectories is resolved: the two lines are actually five trajectories, no two of which intersect.

Once we know the motion along the four rays, we can put arrowheads on them to indicate the direction of motion along the hyperbolas as t increases, since it must be compatible with the motion along the rays — for by continuity, nearby trajectories must have arrowheads pointing in similar directions. The only possibility therefore is the one shown in fig. 2.

A linear system whose trajectories show the general features of those in fig. 2 is said to be an **unstable saddle**. It is called *unstable* because the trajectories go off to infinity as t increases (there are three exceptions: what are they?); it is called a *saddle* because of its general resemblance to the level curves of a saddle-shaped surface in 3-space.

Example 3.2 This time we consider the linear system below — since it is decoupled, its general solution (on the right) can be obtained easily by inspection:

$$(12) \quad \begin{cases} x' = -x \\ y' = -2y \end{cases} \quad \mathbf{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}.$$

Converting it as in (5) to a single first order ODE and solving it by separating variables gives as the general solutions (on the right below) a family of parabolas:

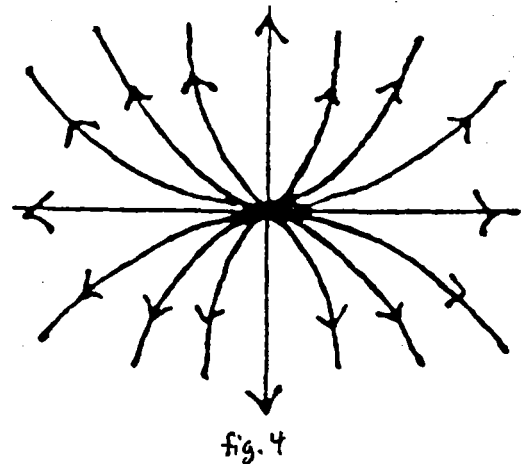
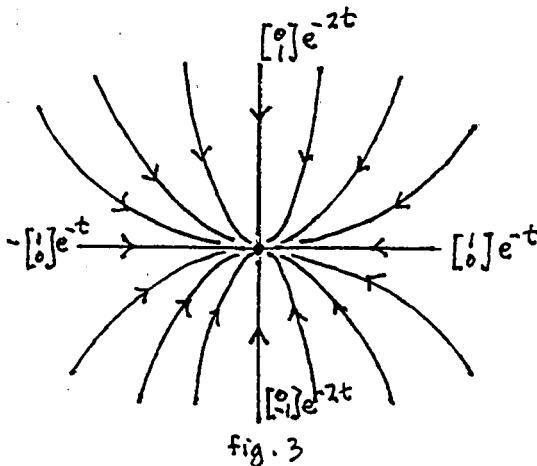
$$\frac{dy}{dx} = \frac{2y}{x}; \quad y = cx^2.$$

Following the same plan as in Example 3.1, we single out the four solutions

$$(13) \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t}, \quad -\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}, \quad -\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}.$$

Their trajectories are the four rays along the coordinate axes, the motion being always inward as t increases. Put compatible arrowheads on the parabolas and you get fig. 3.

A linear system whose trajectories have the general shape of those in fig. 3 is called an **asymptotically stable node** or a **sink node**. The word *node* is used when the trajectories have a roughly parabolic shape (or exceptionally, they are rays); *asymptotically stable* or *sink* means that all the trajectories approach the critical point as t increases.



Example 3.3 This is the same as Example 3.2, except that the signs are reversed:

$$(12) \quad \begin{cases} x' = x \\ y' = 2y \end{cases} \quad \mathbf{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t}.$$

The first-order differential equation remains the same; we get the same parabolas. The only difference in the work is that the exponentials now have positive exponents; the picture remains exactly the same except that now the trajectories are all traversed in the opposite direction — away from the origin — as t increases. The resulting picture is fig. 4, which we call an **unstable node** or **source node**.

Example 3.4 A different type of simple system (eigenvalues $\pm i$) and its solution is

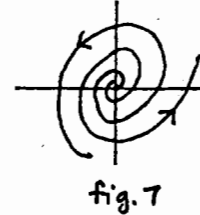
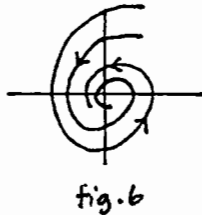
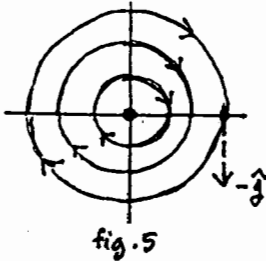
$$(14) \quad \begin{cases} x' = y \\ y' = -x \end{cases} ; \quad \mathbf{x} = c_1 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} .$$

Converting to a first-order ODE by (5) and solving by separation of variables gives

$$\frac{dy}{dx} = -\frac{x}{y}, \quad x^2 + y^2 = c ;$$

the trajectories are the family of circles centered at the origin. To determine the direction of motion, look at the solution in (14) for which $c_1 = 0$, $c_2 = 1$; it is the reflection in the y -axis of the usual (counterclockwise) parametrization of the circle; hence the motion is *clockwise* around the circle. An even simpler procedure is to determine a single vector in the velocity field — that's enough to determine all of the directions. For example, the velocity vector at $(1, 0)$ is $\langle 0, -1 \rangle = -\mathbf{j}$, again showing the motion is clockwise. (The vector is drawn in on fig. 5, which illustrates the trajectories.)

This type of linear system is called a **stable center**. The word *stable* signifies that any trajectory stays within a bounded region of the phase plane as t increases or decreases indefinitely. (We cannot use "asymptotically stable," since the trajectories do not approach the critical point $(0, 0)$ as t increases. The word *center* describes the geometric configuration: it would be used also if the curves were ellipses having the origin as center.)



Example 3.5 As a last example, a system having a complex eigenvalue $\lambda = -1 + i$ is, with its general solution,

$$(15) \quad \begin{cases} x' = -x + y \\ y' = -x - y \end{cases} \quad \mathbf{x} = c_1 e^{-t} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} .$$

The two fundamental solutions (using $c_1 = 0$ and $c_1 = 1$, and vice-versa) are typical. They are like the solutions in Example 3.4, but multiplied by e^{-t} . Their trajectories are therefore traced out by the tip of an origin vector that rotates clockwise at a constant rate, while its magnitude shrinks exponentially to 0: the trajectories spiral in toward the origin as t increases. We call this pattern an **asymptotically stable spiral** or a *sink spiral*; see fig. 6. (An older terminology uses *focus* instead of spiral.)

To determine the direction of motion, it is simplest to do what we did in the previous example: determine from the ODE system a single vector of the velocity field: for instance, the system (15) has at $(1, 0)$ the velocity vector $-\mathbf{i} - \mathbf{j}$, which shows the motion is clockwise.

For the system $\begin{cases} x' = x + y \\ y' = -x + y \end{cases}$, an eigenvalue is $\lambda = 1 + i$, and in (15) e^t replaces e^{-t} ; the magnitude of the rotating vector increases as t increases, giving as pattern an **unstable spiral**, or *source spiral*, as in fig. 7.

4. Sketching more general linear systems.

In the preceding section we sketched trajectories for some particular linear systems; they were chosen to illustrate the different possible geometric pictures. Based on that experience, we can now describe how to sketch the general system

$$\mathbf{x}' = A\mathbf{x}, \quad A = 2 \times 2 \text{ constant matrix.}$$

The geometric picture is largely determined by the eigenvalues and eigenvectors of A , so there are several cases.

For the first group of cases, we suppose the eigenvalues λ_1 and λ_2 are *real* and *distinct*.

Case 1. The λ_i have opposite signs: $\lambda_1 > 0, \lambda_2 < 0$; **unstable saddle.**

Suppose the corresponding eigenvectors are $\vec{\alpha}_1$ and $\vec{\alpha}_2$, respectively. Then four solutions to the system are

$$(16) \quad \mathbf{x} = \pm \vec{\alpha}_1 e^{\lambda_1 t}, \quad \mathbf{x} = \pm \vec{\alpha}_2 e^{\lambda_2 t}.$$

How do the trajectories of these four solutions look?

In fig. 8 below, the four vectors $\pm \vec{\alpha}_1$ and $\pm \vec{\alpha}_2$ are drawn as origin vectors; in fig. 9, the corresponding four trajectories are shown as solid lines, with the direction of motion as t increases shown by arrows on the lines. The reasoning behind this is the following.

Look first at $\mathbf{x} = \vec{\alpha}_1 e^{\lambda_1 t}$. We think of $e^{\lambda_1 t}$ as a scalar factor changing the length of \mathbf{x} ; as t increases from $-\infty$ to ∞ , this scalar factor increases from 0 to ∞ , since $\lambda_1 > 0$. The tip of this lengthening vector represents the trajectory of the solution $\mathbf{x} = \vec{\alpha}_1 e^{\lambda_1 t}$, which is therefore a ray going out from the origin in the direction of the vector $\vec{\alpha}_1$.

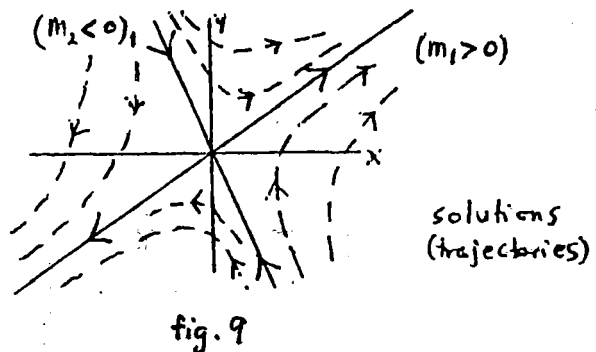
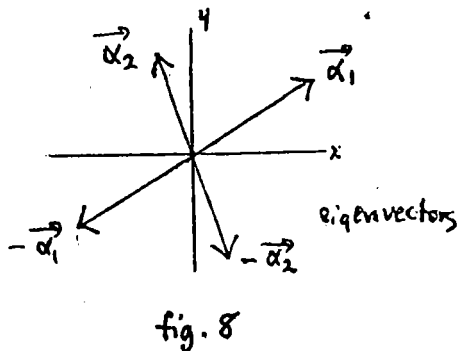
Similarly, the trajectory of $\mathbf{x} = -\vec{\alpha}_1 e^{\lambda_1 t}$ is a ray going out from the origin in the opposite direction: that of the vector $-\vec{\alpha}_1$.

The trajectories of the other two solutions $\mathbf{x} = \pm \vec{\alpha}_2 e^{\lambda_2 t}$ will be similar, except that since $\lambda_2 < 0$, the scalar factor $e^{\lambda_2 t}$ decreases as t increases; thus the solution vector will be shrinking as t increases, so the trajectory traced out by its tip will be a ray having the direction of $\vec{\alpha}_2$ or $-\vec{\alpha}_2$, but traversed toward the origin as t increases, getting arbitrarily close but never reaching it in finite time.

To complete the picture, we sketch in some nearby trajectories; these will be smooth curves generally following the directions of the four rays described above. In Example 3.1 they were hyperbolas; in general they are not, but they look something like hyperbolas, and they do have the rays as asymptotes. They are the trajectories of the solutions

$$(17) \quad \mathbf{x} = c_1 \vec{\alpha}_1 e^{\lambda_1 t} + c_2 \vec{\alpha}_2 e^{\lambda_2 t},$$

for different values of the constants c_1 and c_2 .



Case 2. λ_1 and λ_2 are distinct and negative: say $\lambda_1 < \lambda_2 < 0$;

asymptotically stable (sink) node

Formally, the solutions (16) are written the same way, and we draw their trajectories just as before. The only difference is that now all four trajectories are represented by rays coming in towards the origin as t increases, because both of the λ_i are negative. The four trajectories are represented as solid lines in figure 10, on the next page.

The trajectories of the other solutions (17) will be smooth curves which generally follow the four rays. In the corresponding Example 3.2, they were parabolas; here too they will be parabola-like, but this does not tell us how to draw them, and a little more thought is needed. The parabolic curves will certainly come in to the origin as t increases, but tangent to which of the rays? Briefly, the answer is this:

Node-sketching principle. *Near the origin, the trajectories follow the ray attached to the λ_i nearer to zero; far from the origin, they follow (i.e. are roughly parallel to) the ray attached to the λ_i further from zero.*

You need not memorize the above; instead learn the reasoning on which it is based, since this type of argument will be used over and over in science and engineering work having nothing to do with differential equations.

Since we are assuming $\lambda_1 < \lambda_2 < 0$, it is λ_2 which is closer to 0. We want to know the behavior of the solutions near the origin and far from the origin. Since all solutions are approaching the origin,

near the origin corresponds to large positive t (we write $t \gg 1$):

far from the origin corresponds to large negative t (written $t \ll -1$).

As before, the general solution has the form

$$(18) \quad \mathbf{x} = c_1 \vec{\alpha}_1 e^{\lambda_1 t} + c_2 \vec{\alpha}_2 e^{\lambda_2 t}, \quad \lambda_1 < \lambda_2 < 0.$$

If $t \gg 1$, then \mathbf{x} is near the origin, since both terms in (18) are small; however, the first term is negligible compared with the second: for since $\lambda_1 - \lambda_2 < 0$, we have

$$(19) \quad \frac{e^{\lambda_1 t}}{e^{\lambda_2 t}} = e^{(\lambda_1 - \lambda_2)t} \approx 0, \quad t \gg 1.$$

Thus if $\lambda_1 < \lambda_2 < 0$ and $t \gg 1$, we can neglect the first term of (18), getting

$$\mathbf{x} \sim c_2 \vec{\alpha}_2 e^{\lambda_2 t}. \quad \text{for } t \gg 1 \text{ (x near the origin),}$$

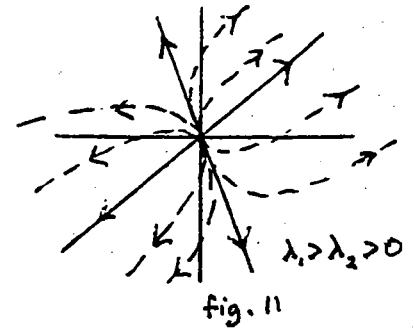
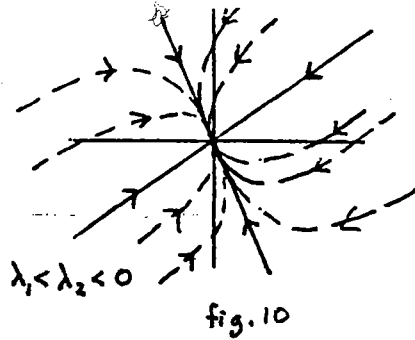
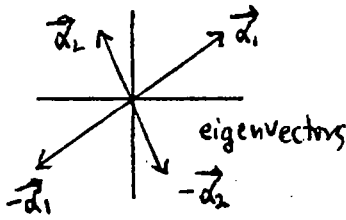
which shows that $\mathbf{x}(t)$ follows the ray corresponding to the the eigenvalue λ_2 closer to zero.

Similarly, if $t \ll -1$, then \mathbf{x} is far from the origin since both terms in (18) are large. This time the ratio in (19) is large, so that it is the first term in (18) that dominates the expression, which tells us that

$$\mathbf{x} \sim c_1 \vec{\alpha}_1 e^{\lambda_1 t}. \quad \text{for } t \ll -1 \text{ (x far from the origin).}$$

This explains the reasoning behind the node-sketching principle in this case.

Some of the trajectories of the solutions (18) are sketched in dashed lines in figure 10, using the node-sketching principle, and assuming $\lambda_1 < \lambda_2 < 0$.



Case 3. λ_1 and λ_2 are distinct and positive: say $\lambda_1 > \lambda_2 > 0$ **unstable (source) node**

The analysis is like the one we gave above. The direction of motion on the four rays coming from the origin is outwards, since the $\lambda_i > 0$. The node-sketching principle is still valid, and the reasoning for it is like the reasoning in case 2. The resulting sketch looks like the one in fig. 11.

Case 4. Eigenvalues pure imaginary: $\lambda = \pm bi$, $b > 0$ **stable center**

Here the solutions to the linear system have the form

$$(20) \quad \mathbf{x} = \mathbf{c}_1 \cos bt + \mathbf{c}_2 \sin bt, \quad \mathbf{c}_1, \mathbf{c}_2 \text{ constant vectors.}$$

(There is no exponential factor since the real part of λ is zero.) Since every solution (20) is periodic, with period $2\pi/b$, the moving point representing it retraces its path at intervals of $2\pi/b$. The trajectories therefore are closed curves; ellipses, in fact; see fig. 12.

Sketching the ellipse is a little troublesome, since the vectors \mathbf{c}_i do not have any simple relation to the major and minor axes of the ellipse. For this course, it will be enough if you determine whether the motion is clockwise or counterclockwise. As in Example 3.4, this can be done by using the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ to calculate a single velocity vector \mathbf{x}' of the velocity field; from this the sense of motion can be determined by inspection.

The word *stable* means that each trajectory stays for all time within some circle centered at the critical point; *asymptotically stable* is a stronger requirement: each trajectory must approach the critical point (here, the origin) as $t \rightarrow \infty$.

Case 5. The eigenvalues are complex, but not purely imaginary; there are two cases:

$a \pm bi$, $a < 0$, $b > 0$; **asymptotically stable (sink) spiral;**

$a \pm bi$, $a > 0$, $b > 0$; **unstable (source) spiral;**

Here the solutions to the linear system have the form

$$(21) \quad \mathbf{x} = e^{at}(\mathbf{c}_1 \cos bt + \mathbf{c}_2 \sin bt), \quad \mathbf{c}_1, \mathbf{c}_2 \text{ constant vectors.}$$

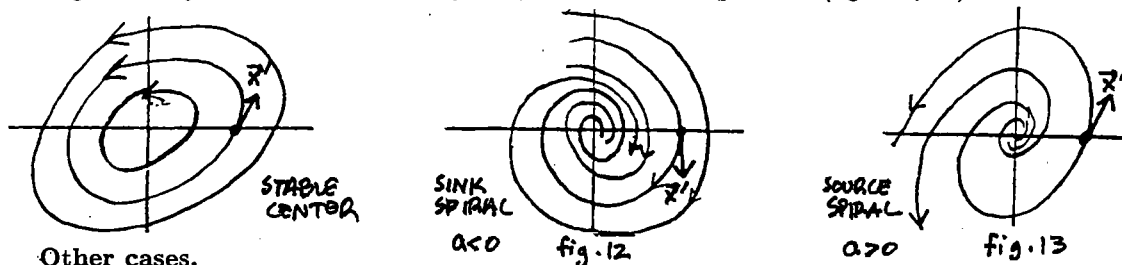
They look like the solutions (20), except for a scalar factor e^{at} which either

decreases towards 0 as $t \rightarrow \infty$ ($a < 0$), or
increases towards ∞ as $t \rightarrow \infty$ ($a > 0$).

Thus the point \mathbf{x} travels in a trajectory which is like an ellipse, except that the distance from the origin keeps steadily shrinking or expanding. The result is a trajectory which does one of the following:

spirals steadily towards the origin, (asymptotically stable spiral): $a < 0$
 spirals steadily away from the origin. (unstable spiral); $a > 0$

The exact shape of the spiral, is not obvious and perhaps best left to computers; but you should determine the direction of motion, by calculating from the linear system $\mathbf{x}' = A\mathbf{x}$ a single velocity vector \mathbf{x}' near the origin. Typical spirals are pictured (figs. 12, 13).



Other cases.

Repeated real eigenvalue $\lambda \neq 0$, defective (one independent eigenvector)
defective node; unstable if $\lambda > 0$; asymptotically stable if $\lambda < 0$ (fig. 14)

Repeated real eigenvalue $\lambda \neq 0$, complete (two independent eigenvectors)
star node; unstable if $\lambda > 0$; asymptotically stable if $\lambda < 0$. (fig. 15)

One eigenvalue $\lambda = 0$. (Picture left for exercises and problem sets.)

5. Summary

To sum up, the procedure of sketching trajectories of the 2×2 linear homogeneous system $\mathbf{x}' = A\mathbf{x}$, where A is a constant matrix, is this. Begin by finding the eigenvalues of A .

1. If they are real, distinct, and non-zero:

- find the corresponding eigenvectors;
- draw in the corresponding solutions whose trajectories are rays; use the sign of the eigenvalue to determine the direction of motion as t increases; indicate it with an arrowhead on the ray;
- draw in some nearby smooth curves, with arrowheads indicating the direction of motion:

- if the eigenvalues have opposite signs, this is easy;
- if the eigenvalues have the same sign, determine which is the dominant term in the solution for $t \gg 1$ and $t \ll -1$, and use this to determine which rays the trajectories are tangent to, near the origin, and which rays they are parallel to, away from the origin. (Or use the node-sketching principle.)

2. If the eigenvalues are complex: $a \pm bi$, the trajectories will be
 ellipses if $a = 0$,
 spirals if $a \neq 0$: inward if $a < 0$, outward if $a > 0$;

in all cases, determine the direction of motion by using the system $\mathbf{x}' = A\mathbf{x}$ to find one velocity vector.

3. The details in the other cases (eigenvalues repeated, or zero) will be left as exercises using the reasoning in this section.

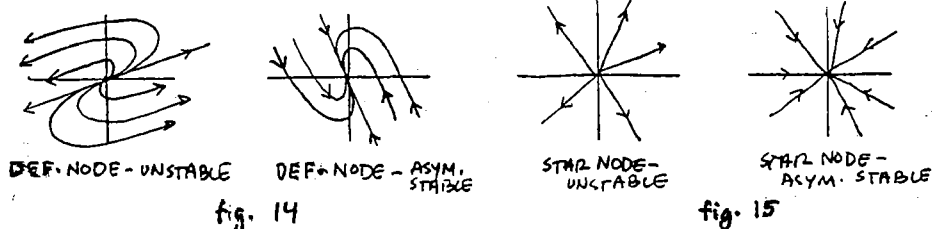


fig. 14

fig. 15

6. Sketching non-linear systems

In sections 3, 4, and 5, we described how to sketch the trajectories of a linear system

$$\begin{aligned}x' &= ax + by \\y' &= cx + dy\end{aligned}\quad a, b, c, d \text{ constants.}$$

We now return to the general (i.e., non-linear) 2×2 autonomous system discussed at the beginning of this chapter, in sections 1 and 2:

$$(22) \quad \begin{aligned}x' &= f(x, y) \\y' &= g(x, y)\end{aligned};$$

it is represented geometrically as a vector field, and its trajectories — the solution curves — are the curves which at each point have the direction prescribed by the vector field. Our goal is to see how one can get information about the trajectories of (22), without determining them analytically or using a computer to plot them numerically.

Linearizing at the origin. To illustrate the general idea, let's suppose that $(0, 0)$ is a critical point of the system (22), i.e.,

$$(23) \quad f(0, 0) = 0, \quad g(0, 0) = 0,$$

Then if f and g are sufficiently differentiable, we can approximate them near $(0, 0)$ (the approximation will have no constant term by (23)):

$$\begin{aligned}f(x, y) &= a_1x + b_1y + \text{higher order terms in } x \text{ and } y \\g(x, y) &= a_2x + b_2y + \text{higher order terms in } x \text{ and } y.\end{aligned}$$

If (x, y) is close to $(0, 0)$, then x and y will be small and we can neglect the higher order terms. Then the non-linear system (23) is approximated near $(0, 0)$ by a linear system, the **linearization** of (23) at $(0, 0)$:

$$(24) \quad \begin{aligned}x' &= a_1x + b_1y \\y' &= a_2x + b_2y\end{aligned},$$

and near $(0, 0)$, the solutions of (22) — about which we know nothing — will be like the solutions to (24), about which we know a great deal from our work in the previous sections.

Example 6.1 Linearize the system $\begin{cases} x' = y \cos x \\ y' = x(1 + y)^2 \end{cases}$ at the critical point $(0, 0)$.

Solution We have $\begin{cases} x' \approx y(1 - \frac{1}{2}x^2) \\ y' = x(1 + 2y + y^2) \end{cases}$ so the linearization is $\begin{cases} x' = y \\ y' = x \end{cases}$.

Linearising at a general point More generally, suppose now the critical point of (22) is (x_0, y_0) , so that

$$f(x_0, y_0) = 0, \quad g(x_0, y_0) = 0.$$

One way this can be handled is to make the change of variable

$$(24) \quad x_1 = x - x_0, \quad y_1 = y - y_0;$$

in the x_1y_1 -coordinate system, the critical point is $(0, 0)$, and we can proceed as before.

Example 6.2 Linearize $\begin{cases} x' = x - x^2 - 2xy \\ y' = y - y^2 - \frac{3}{2}xy \end{cases}$ at its critical points on the x -axis.

Solution. When $y = 0$, the functions on the right are zero when $x = 0$ and $x = 1$, so the critical points on the x -axis are $(0, 0)$ and $(1, 0)$.

The linearization at $(0, 0)$ is $x' = x, \quad y' = y$.

To find the linearization at $(1, 0)$ we change of variable as in (24): $x_1 = x - 1, \quad y_1 = y$; substituting for x and y in the system and keeping just the linear terms on the right gives us as the linearization:

$$\begin{aligned} x_1' &= (x_1 + 1) - (x_1 + 1)^2 - 2(x_1 + 1)y_1 \approx -x_1 - 2y_1 \\ y_1' &= y_1 - y_1^2 - \frac{3}{2}(x_1 + 1)y_1 \approx -\frac{1}{2}y_1. \end{aligned}$$

Linearization using the Jacobian matrix

Though the above techniques are usable if the right sides are very simple, it is generally faster to find the linearization by using the Jacobian matrix, especially if there are several critical points, or the functions on the right are not simple polynomials. We derive the procedure.

We need to approximate f and g near (x_0, y_0) . While this can sometimes be done by changing variable, a more basic method is to use the main approximation theorem of multivariable calculus. For this we use the notation

$$(25) \quad \Delta x = x - x_0, \quad \Delta y = y - y_0, \quad \Delta f = f(x, y) - f(x_0, y_0)$$

and we have then the basic approximation formula

$$\begin{aligned} \Delta f &\approx \left(\frac{\partial f}{\partial x}\right)_0 \Delta x + \left(\frac{\partial f}{\partial y}\right)_0 \Delta y, \quad \text{or} \\ (26) \quad f(x, y) &\approx \left(\frac{\partial f}{\partial x}\right)_0 \Delta x + \left(\frac{\partial f}{\partial y}\right)_0 \Delta y, \end{aligned}$$

since by hypothesis $f(x_0, y_0) = 0$. We now make the change of variables (24)

$$x_1 = x - x_0 = \Delta x, \quad y_1 = y - y_0 = \Delta y,$$

and use (26) to approximate f and g by their linearizations at (x_0, y_0) . The result is that in the neighborhood of the critical point (x_0, y_0) , the linearization of the system (22) is

$$(27) \quad \begin{aligned} x_1' &= \left(\frac{\partial f}{\partial x}\right)_0 x_1 + \left(\frac{\partial f}{\partial y}\right)_0 y_1, \\ y_1' &= \left(\frac{\partial g}{\partial x}\right)_0 x_1 + \left(\frac{\partial g}{\partial y}\right)_0 y_1. \end{aligned}$$

In matrix notation, the linearization is therefore

$$(28) \quad \mathbf{x}'_1 = A \mathbf{x}_1, \quad \text{where } \mathbf{x}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \text{ and } A = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}_{(x_0, y_0)};$$

the matrix A is the Jacobian matrix, evaluated at the critical point (x_0, y_0) .

General procedure for sketching the trajectories of non-linear systems.

We can now outline how to sketch in a qualitative way the solution curves of a 2×2 non-linear autonomous system,

$$(29) \quad \begin{aligned} x' &= f(x, y) \\ y' &= g(x, y). \end{aligned}$$

1. Find all the critical points (i.e., the constant solutions), by solving the system of simultaneous equations

$$\begin{aligned} f(x, y) &= 0 \\ g(x, y) &= 0. \end{aligned}$$

2. For each critical point (x_0, y_0) , find the matrix A of the linearized system at that point, by evaluating the Jacobian matrix at (x_0, y_0) :

$$\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}_{(x_0, y_0)}$$

(Alternatively, make the change of variables $x_1 = x - x_0$, $y_1 = y - y_0$, and drop all terms having order higher than one; then A is the matrix of coefficients for the linear terms.)

3. Find the geometric type and stability of the linearized system at the critical point (x_0, y_0) , by carrying out the analysis in sections 4 and 5.

The subsequent steps require that the eigenvalues be non-zero, real, and distinct, or complex, with a non-zero real part. The remaining cases: eigenvalues which are zero, repeated, or pure imaginary are classified as borderline, and the subsequent steps don't apply, or have limited application. See the next section.

4. According to the above, the acceptable geometric types are a saddle, node (not a star or a defective node, however), and a spiral. Assuming that this is what you have, for each critical point determine enough additional information (eigenvectors, direction of motion) to allow a sketch of the trajectories near the critical point.

5. In the xy -plane, mark the critical points. Around each, sketch the trajectories in its immediate neighborhood, as determined in the previous step, including the direction of motion.

6. Finally, sketch in some other trajectories to fill out the picture, making them compatible with the behavior of the trajectories you have already sketched near the critical points. Mark with an arrowhead the direction of motion on each trajectory.

If you have made a mistake in analyzing any of the critical points, it will often show up here — it will turn out to be impossible to draw in any plausible trajectories that complete the picture.

Remarks about the steps.

1. In the homework problems, the simultaneous equations whose solutions are the critical points will be reasonably easy to solve. In the real world, they may not be; a

2. If there are several critical points, one almost always uses the Jacobian matrix; if there is only one, use your judgment.

3. This method of analyzing non-linear systems rests on the assumption that in the neighborhood of a critical point, the non-linear system will look like its linearization at that point. For the borderline cases this may not be so — that is why they are rejected. The next two sections explain this more fully.

If one or more of the critical points turn out to be borderline cases, one usually resorts to numerical computation on the non-linear system. Occasionally one can use the reduction (section 2) to a first-order equation:

$$\frac{dy}{dx} = \frac{g(x, y)}{f(x, y)}$$

to get information about the system.

Example 6.3 Sketch some trajectories of the system
$$\begin{cases} x' = -x + xy \\ y' = -2y + xy \end{cases}$$

Solution. We first find the critical points, by solving
$$\begin{cases} -x + xy = x(-1 + y) = 0 \\ -2y + xy = y(-2 + x) = 0 \end{cases}$$

From the first equation, either $x = 0$ or $y = 1$. From the second equation,

$$x = 0 \Rightarrow y = 0; \quad y = 1 \Rightarrow x = 2; \quad \text{critical points: } (0, 0), (2, 1).$$

To linearize at the critical points, we compute the Jacobian matrices

$$J = \begin{pmatrix} -1+y & x \\ y & -2+x \end{pmatrix}; \quad J_{(0,0)} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \quad J_{(2,1)} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}.$$

Analyzing the geometric type and stability of each critical point:

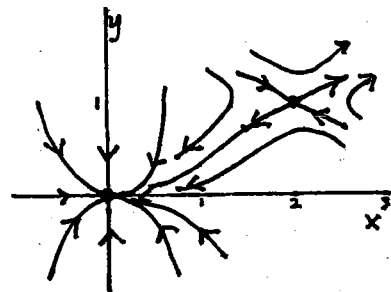
$$\begin{aligned} (0, 0): \text{ eigenvalues: } \lambda_1 = -1, \quad \lambda_2 = -2 \quad \text{sink node} \\ \text{eigenvectors: } \vec{\alpha}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \vec{\alpha}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

By the node-sketching principle, trajectories follow $\vec{\alpha}_1$ near the origin, are parallel to $\vec{\alpha}_2$ away from the origin.

$$\begin{aligned} (2, 1): \text{ eigenvalues: } \lambda_1 = \sqrt{2}, \quad \lambda_2 = -\sqrt{2} \quad \text{unstable saddle} \\ \text{eigenvectors: } \vec{\alpha}_1 = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}; \quad \vec{\alpha}_2 = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} \end{aligned}$$

Draw in these eigenvectors at the respective points $(0, 0)$ and $(2, 1)$, with arrowhead indicating direction of motion (into the critical point if $\lambda < 0$, away from critical point if $\lambda > 0$.) Draw in some nearby trajectories.

Then guess at some other trajectories compatible with these. See the figure for one attempt at this. Further information could be gotten by considering the associated first-order ODE in x and y .



7. Structural Stability

In the previous two sections, we described how to get a rough picture of the trajectories of a non-linear system by linearizing at each of its critical points. The basic assumption of the method is that the linearized system will be a good approximation to the original non-linear system if you stay near the critical point.

The method only works however if the linearized system turns out to be a node, saddle, or spiral. What is it about these geometric types that allows the method to work, and why won't it work if the linearized system turns out to be one of the other possibilities (dismissed as "borderline types" in the previous section)?

Briefly, the answer is that nodes, saddles, and spirals are *structurally stable*, while the other possibilities are not. We call a system

$$(1) \quad \begin{aligned} x' &= f(x, y) \\ y' &= g(x, y) \end{aligned}$$

structurally stable if small changes in its parameters (i.e., the constants that enter into the functions on the right hand side) do not change the geometric type and stability of its critical points (or its limit cycles, if there are any — see the next section, and don't worry about them for now.)

Theorem. *The 2×2 autonomous linear system*

$$(2) \quad \begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}$$

is structurally stable if it is a spiral, saddle, or node (but not a degenerate or star node).

Proof. The characteristic equation is

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0,$$

and its roots (the eigenvalues) are

$$(3) \quad \lambda_1, \lambda_2 = \frac{(a + d) \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2}.$$

Let's look at the cases one-by-one; assume first that the roots λ_1 and λ_2 are real and distinct. The possibilities in the theorem are given by the following (note that since the roots are distinct, the node will not be degenerate or a star node):

$$\begin{aligned} \lambda_1 > 0, \lambda_2 > 0 & \text{ unstable node} \\ \lambda_1 < 0, \lambda_2 < 0 & \text{ asymptotically stable node} \\ \lambda_1 > 0, \lambda_2 < 0 & \text{ unstable saddle.} \end{aligned}$$

The quadratic formula (3) shows that the roots depend continuously on the coefficients a, b, c, d . Thus if the coefficients are changed a little, the roots λ_1 and λ_2 will also be changed a little to λ'_1 and λ'_2 respectively; the new roots will still be real, and will have the same sign if the change is small enough. Thus the changed system will still have the same geometric type and stability. \square

If the roots of the characteristic equation are complex, the reasoning is similar. Let us denote the complex roots by $r \pm si$; we use the root $\lambda = r + si$, $s > 0$; then the possibilities to be considered for structural stability are

$$r > 0, s > 0 \quad \text{unstable spiral}$$

$$r < 0, s > 0 \quad \text{asymptotically stable spiral.}$$

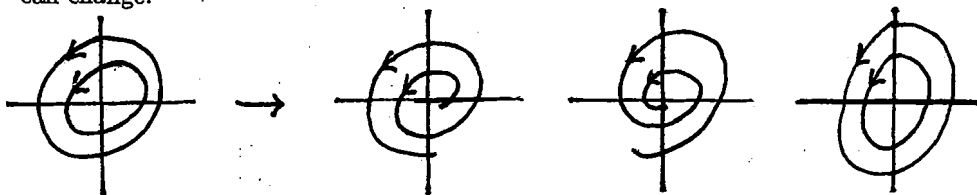
If a, b, c, d are changed a little, the root is changed to $\lambda' = r' + s'i$, where r' and s' are close to r and s respectively, since the quadratic formula (3) shows r and s depend continuously on the coefficients. If the change is small enough, r' will have the same sign as r and s' will still be positive, so the geometric type of the changed system will still be a spiral, with the same stability type. $\square\square$

8. The borderline geometric types All the other possibilities for the linear system (2) we call borderline types. We will show now that none of them is structurally stable; we begin with the center.

Eigenvalues pure imaginary. Once again we use the eigenvalue with the positive imaginary part: $\lambda = 0 + si$, $s > 0$. It corresponds to a *center*: the trajectories are a family of concentric ellipses, centered at the origin. If the coefficients a, b, c, d are changed a little, the eigenvalue $0 + si$ changes a little to $r' + s'i$, where $r' \approx 0$, $s' \approx s$, and there are three possibilities for the new eigenvalue:

$0 + si \rightarrow r' + s'i:$	$r' > 0$	$r' < 0$	$r' = 0$
	$s' > 0$	$s' > 0$	$s' > 0$
<i>center</i>	<i>source spiral</i>	<i>sink spiral</i>	<i>center</i>

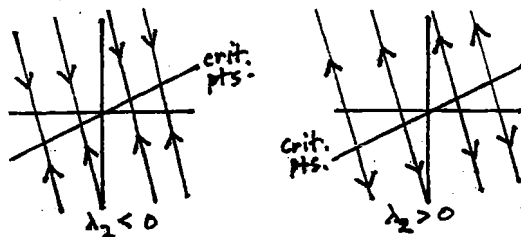
Correspondingly, there are three possibilities for how the geometric picture of the trajectories can change:



Eigenvalues real; one eigenvalue zero. Here $\lambda_1 = 0$, and $\lambda_2 > 0$ or $\lambda_2 < 0$. The general solution to the system has the form (α_1, α_2 are the eigenvectors)

$$x = c_1\alpha_1 + c_2\alpha_2e^{\lambda_2 t}.$$

If $\lambda_2 < 0$, the geometric picture of its trajectories shows a line of critical points (constant solutions, corresponding to $c_2 = 0$), with all other trajectories being parallel lines ending up (for $t = \infty$) at one of the critical points, as shown below.

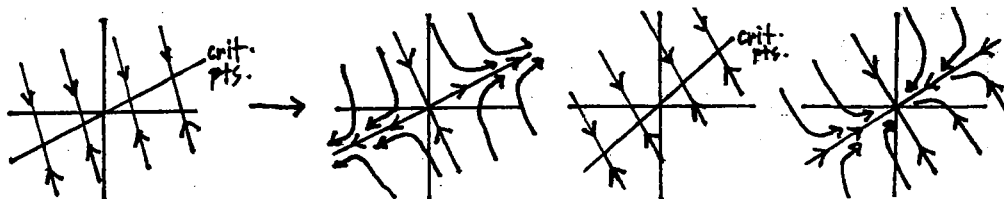


If $\lambda_2 > 0$, the picture is the same, with the arrows reversed.

We continue to assume $\lambda_2 < 0$. As the coefficients of the system change a little, the two eigenvalues change a little also; there are three possibilities, since the 0 eigenvalue can become positive, negative, or stay zero:

$\lambda_1 = 0 \rightarrow \lambda'_1 :$	$\lambda'_1 > 0$	$\lambda'_1 = 0$	$\lambda_1 < 0$
$\lambda_2 < 0 \rightarrow \lambda'_2 :$	$\lambda'_2 < 0$	$\lambda'_2 < 0$	$\lambda'_2 < 0$
<i>critical line</i>	<i>unstable saddle</i>	<i>critical line</i>	<i>sink node</i>

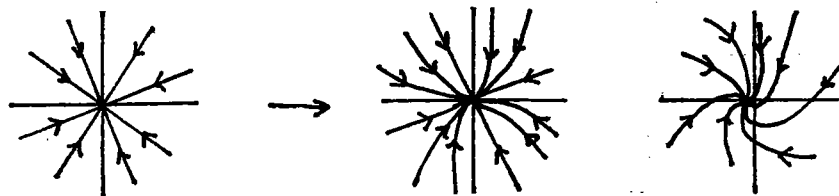
Here are the corresponding pictures. (The pictures would look the same if we assumed $\lambda_2 > 0$, but the arrows on the trajectories would be reversed.)



One repeated real eigenvalue. Finally, we consider the case where $\lambda_1 = \lambda_2$. Here there are a number of possibilities, depending on whether λ_1 is positive or negative, and whether the repeated eigenvalue is complete (i.e., has two independent eigenvectors), or defective (i.e., incomplete: only one eigenvector). Let us assume that $\lambda_1 < 0$. We vary the coefficients of the system a little. By the same reasoning as before, the eigenvalues change a little, and by the same reasoning as before, we get as the main possibilities (omitting this time the one where the changed eigenvalue is still repeated):

$\lambda_1 < 0$	\rightarrow	$\lambda'_1 < 0$	$r + si$
$\lambda_2 < 0$	\rightarrow	$\lambda'_2 < 0$	$r - si$
$\lambda_1 = \lambda_2$		$\lambda'_1 \neq \lambda'_2$	$r \approx \lambda_1, s \approx 0,$
<i>sink node</i>		<i>sink node</i>	<i>sink spiral</i>

Typical corresponding pictures for the complete case and the defect case are (the last one is left for you to experiment with on the computer screen)



complete: star node

incomplete: defective node

Remarks. Each of these three cases—one eigenvalue zero, pure imaginary eigenvalues, repeated real eigenvalue—has to be looked on as a borderline linear system: altering the coefficients slightly can give it an entirely different geometric type, and in the first two cases, possibly alter its stability as well.

Application to non-linear systems.

All the preceding analysis discussed the structural stability of a linear system. How does it apply to non-linear systems?

Suppose our non-linear system has a critical point at P , and we want to study its trajectories near P by linearizing the system at P .

This linearization is only an approximation to the original system, so if it turns out to be a borderline case, i.e., one sensitive to the exact value of the coefficients, *the trajectories near P of the original system can look like any of the types obtainable by slightly changing the coefficients of the linearization.*

In other words, the method of linearization we used in Sections 6 and 7 to analyze a non-linear system near a critical point doesn't fail entirely, but we don't end up with a definite picture of the non-linear system near P ; we only get a list of possibilities. In general one has to rely on computation or more powerful analytic tools to get a clearer answer.