IR. Input-Response Models

1. First-order linear ODE's. If we use t as the independent variable, thinking of it as representing time, the linear first-order IVP in standard linear form is

$$\frac{dy}{dt} + p(t)y = q(t), \qquad y(0) = y_0. \tag{1}$$

For the most part, this section will be about the most important special case: the *constant coefficient* case, where $p(t)$ is a constant function k:

$$\frac{dy}{dt} + ky = q(t), \qquad y(0) = y_0. \tag{2}$$

The integrating factor for the ODE (2) is e^{kt}; using it, the general solution is

$$y = e^{-kt}\left(\int q(t)e^{kt}\,dt + c\right). \tag{3}$$

Since the integral is only determined up to an arbitrary constant, (3) as it stands does not give an explicit solution to the IVP (2). To get this, we use instead (cf. Notes D) a definite integral from 0 to t, which requires us to change the t in the integrand to a different dummy variable, u say; then the explicit solution to the IVP (2) is

$$y = e^{-kt}\int_0^t q(u)e^{ku}\,du + y_0 e^{-kt}. \tag{4}$$

In this form, note that the first term on the right is the solution to the IVP (2) with the initial condition $y_0 = 0$.

Terminology when k is positive. If $k > 0$, the **long-term** or **steady-state** solution is what we call the first term on the right side of (4); it does not involve the initial value y_0. The second term on the right of (4), which does involve y_0, is called the **transient** since its effect dies away as time increases: $\lim_{t\to\infty} y_0 e^{-kt} = 0$. Thus, if k is positive, all solutions (4) to the ODE (2) tend toward the steady-state solution, as $t \to \infty$.

Despite the use of the definite article, the steady-state solution is not unique: since all the solutions approach the steady-state solution as $t \to \infty$, they all approach each other, and thus any of them can be taken as the steady-state solution. In practice, it is usually the simplest-looking solution which is given this honor.

2. Input-response; superposition of inputs.

When the ODE (1) is used to model a physical situation, the left-hand side usually is concerned with the physical set-up — the "system" — while the right-hand side represents something external which is driving or otherwise affecting the system from the outside. For this reason, the function $q(t)$ is often called the **input**, or in some engineering subjects, the **signal**; the corresponding solution $y(t)$ is generally called the **response** of the system to this input. The response will be a definite function if an initial condition has been specified, otherwise it will be the general solution, involving an arbitrary constant. We will indicate the relation of input to response symbolically by

$$q(t) \rightsquigarrow y(t) \qquad (\textit{input} \rightsquigarrow \textit{response}).$$

Superposition principle for inputs.

For the ODE $y' + p(t)y = q(t)$, *let* q_1 *and* q_2 *be inputs, and* c_1, c_2 *constants. Then*

$$(5) \qquad q_1 \rightsquigarrow y_1, \quad q_2 \rightsquigarrow y_2 \quad \Rightarrow \quad c_1q_1 + c_2q_2 \rightsquigarrow c_1y_1 + c_2y_2.$$

Proof. This is true because the ODE (1) is linear. The proof takes one line:

$$(c_1y_1 + c_2y_2)' + p(c_1y_1 + c_2y_2) = (c_1y_1' + pc_1y_1) + (c_2y_2' + pc_2y_2) = c_1q_1 + c_2q_2.$$

The superposition principle allows us to break up a problem into simpler problems and then at the end assemble the answer from its simpler pieces. Here is an easy example.

Example. Find the response of $y' + 2y = q(t)$ to $q = 1 + e^{-2t}$.

Solution. The input $q = 1$ generates the response $y = 1/2$, by inspection; the input e^{-2t} generates the response te^{-2t}, by solving; therefore the response to $1 + e^{-2t}$ is $1/2 + te^{-2t}$. □

3. When k is positive and constant: system responses to linear inputs.

We return once again to the case when $p(t)$ is a constant $k > 0$. We want to get some feeling for how the system response is related to the input. The temperature model will be a good guide: in two notations – suggestive and neutral, respectively – the ODE is

$$(6) \qquad T' + kT = kT_e(t), \qquad\qquad y' + ky = kq_e(t) = q(t).$$

Note that the neutral notation writes the input in two different forms: the $q(t)$ we have been using, and also in the form $kq_e(t)$ with the k factored out. This corresponds to the way the input normally appears in physical problems and offers some advantages: for instance, q_e and y have the same units, whereas q and y do not. In trying to relate response with input, the relation will be clearer if we relate y with q_e, rather than with q. We will use for q_e the generic name **physical input**, or if we have a specific model in mind, the *temperature* input, *concentration* input, and so on.

The expected behavior of the temperature model suggests general questions such as:
Is the response the same type of function as the physical input?
What controls its size?
Does the graph of the response lag behind that of the physical input?
What controls the size of the lag?

Our plan will be to get some feeling for the situation by answering these questions for several simple physical inputs. Throughout, keep the temperature model in mind to guide your intuition.

Example 2. Find the response of the system (6) to the physical inputs 1 and t.

Solution. The ODE is $y' + ky = kq_e$.
If $q_e = 1$, a solution by inspection is $y = 1$, so the response is 1.

If $q_e = t$, the ODE is $y' + ky = kt$; using the integrating factor e^{kt} and subsequent integration by parts leads (cf. (3)) to the simplest steady-state solution

$$\begin{aligned} y &= e^{-kt} \int kte^{kt}dt \\ &= k\,e^{-kt}\left(\frac{te^{kt}}{k} - \frac{e^{kt}}{k^2}\right) \\ &= t - \frac{1}{k}. \end{aligned}$$

Thus the response of (6) is identical to the physical input t, but with a time lag $1/k$. This is reasonable when one thinks of the temperature model: the internal temperature increases linearly at the same rate as the temperature of the external water bath, but with a time lag dependent on the conductivity: the higher the conductivity, the shorter the time lag.

Using the superposition principle for inputs, it follows from Example 2 that for the ODE $y' + ky = kq_e$, its response to a general linear physical input is given by:

(7) **linear input** physical input: $q_e = a + bt$ response: $a + b\left(t - \dfrac{1}{k}\right)$.

In the previous example, we paid no attention to initial values. If they are important, one cannot just give the steady-state solution as the response, one has to take account of them, either by using a definite integral as in (4), or by giving the value of the arbitrary constant in (3). Examples in the next section will illustrate.

4. Response to discontinuous inputs, $k > 0$.

The most basic discontinuous function is the unit-step function at a point, defined by

(8) $$u_a(t) = \begin{cases} 0, & t < a; \\ 1, & t > a. \end{cases}$$ **unit-step function at a**

(We leave its value at a undefined, though some books give it the value 0, others the value 1 there.)

Example 3. Find the response of the IVP $y' + ky = kq_e$, $y(0) = 0$, for $t \geq 0$, to the unit-step physical input $u_a(t)$, where $a \geq 0$.

Solution. For $t < a$ the input is 0, so the response is 0. For $t \geq a$, the steady-state solution for the physical input $u_a(t)$ is the constant function 1, according to Example 2 or (7).

We still need to fit the value $y(a) = 0$ to the response for $t \geq a$. Using (3) to do this, we get $1 + ce^{-ka} = 0$, so that $c = -e^{ka}$. We now assemble the results for $t < a$ and $t \geq a$ into one expression; for the latter we also put the exponent in a more suggestive form. We get finally

unit-step input

(9) physical input: $u_a(t),\ a \geq 0$ response: $$y(t) = \begin{cases} 0, & 0 \leq t < a; \\ 1 - e^{-k(t-a)}, & t \geq a. \end{cases}$$

Note that the response is just the translation a units to the right of the response to the unit-step input at 0.

Another way of getting the same answer would be to use the definite integral in (4); we leave this as an exercise.

As another example of discontinuous input, we focus on the temperature model, and obtain the response to the temperature input corresponding to the external bath initially ice-water at 0 degrees, then replaced by water held at a fixed temperature for a time interval, then replaced once more by the ice-water bath.

Example 4. Find the response of $y' + ky = kq_e$ to the physical input

$$u_{ab} = \begin{cases} 1, & a \le t \le b; \\ 0, & \text{otherwise}, \end{cases} \qquad 0 \le a < b; \qquad \textbf{unit-box function} \text{ on } [a,b]. \tag{10}$$

Solution. There are at least three ways to do this:

a) Express u_{ab} as a sum of unit step functions and use (9) together with superposition of inputs;

b) Use the function u_{ab} directly in the definite integral expression (4) for the response;

c) Find the response in two steps: first use (9) to get the response $y(t)$ for the physical input $u_a(t)$; this will be valid up to the point $t = b$.

Then, to continue the response for values $t > b$, evaluate $y(b)$ and find the response for $t > b$ to the input 0, with initial condition $y(b)$.

We will follow (c), leaving the first two as exercises.

By (9), the response to the physical input $u_a(t)$ is $y(t) = \begin{cases} 0, & 0 \le t < a; \\ 1 - e^{-k(t-a)}, & t \ge a. \end{cases}$; this is valid up to $t = b$, since $u_{ab}(t) = u_a(t)$ for $t \le b$. Evaluating at b,

$$y(b) = 1 - e^{-k(b-a)}. \tag{11}$$

Using (3) to find the solution for $t \ge b$, we note first that the steady-state solution will be 0, since $u_{ab} = 0$ for $t > b$; thus by (3) the solution for $t > b$ will have the form

$$y(t) = 0 + ce^{-kt} \tag{12}$$

where c is determined from the initial value (11). Equating the initial values $y(b)$ from (11) and (12), we get

$$ce^{-kb} \;=\; 1 - e^{-kb+ka}$$

from which

$$c = e^{kb} - e^{ka};$$

so by (12),

$$y(t) = (e^{kb} - e^{ka})e^{-kt}, \quad t \ge b. \tag{13}$$

After combining exponents in (13) to give an alternative form for the response, we assemble the parts, getting the *response to the physical unit-box input* u_{ab} :

$$y(t) = \begin{cases} 0, & 0 \le t \le a; \\ 1 - e^{-k(t-a)}, & a < t < b; \\ e^{-k(t-b)} - e^{-k(t-a)}, & t \ge b. \end{cases} \tag{14}$$

5. Response to sinusoidal inputs.

Of great importance in the applications is the sinusoidal input, i.e., a pure oscillation like $\cos\omega t$ or $\sin\omega t$, or more generally, $A\cos(\omega t-\phi)$. (The last form includes both of the previous two, as you can see by letting $A=1$ and $\phi=0$ or $\pi/2$).

In the temperature model, this could represent the diurnal varying of outside temperature; in the conentration model, the diurnal varying of the level of some hormone in the bloodstream, or the varying concentration in a sewer line of some waste product produced periodically by a manufacturing process.

Later on in the term, we shall see that any periodic input $f(t)$ can be represented as an infinite sum (Fourier series) of sinusoidal inputs of different frequencies; then by the superposition principle, we will be able get the response to the periodic input $f(t)$ by adding up the separate responses to each of the sinusoidal inputs.

Response of $y'+ky=kq_e$ to the physical inputs $\cos\omega t$, $\sin\omega t$.

This calculation is a good example of how the use of complex exponentials can simplify integrations and lead to a more compact and above all more expressive answer. You should study it very carefully, since the ideas in it will frequently recur.

We begin by complexifying the inputs, the response, and the differential equation:

$$\cos\omega t = \mathrm{Re}\,(e^{i\omega t}), \qquad \sin\omega t = \mathrm{Im}\,(e^{i\omega t}); \tag{15}$$

$$\tilde{y}(t) = y_1(t)+iy_2(t); \tag{16}$$

$$\tilde{y}'+k\tilde{y} = ke^{i\omega t}. \tag{17}$$

If (16) is a solution to the complex ODE (17), then substituting it into the ODE and using the rule $(u+iv)'=u'+iv'$ for differentiating complex functions (see Notes C, (19)),

$$y_1'+iy_2'+k(y_1+iy_2) = k(\cos\omega t+i\sin\omega t);$$

equating real and imaginary parts on the two sides gives

$$y_1'+ky_1=k\cos\omega t, \qquad y_2'+ky_2=k\sin\omega t; \tag{18}$$

this shows that the real and imaginary parts of our complex solution $\tilde{y}(t)$ give us respectively the solution of the two ODE's in (18).

To solve (17), the integrating factor is e^{kt}; multiplying through by it leads to

$$(\tilde{y}e^{kt})' = ke^{(k+i\omega)t};$$

integrate both sides, multiply through by e^{-kt}, and scale the coefficient to lump constants and make it look better:

$$\tilde{y} = \frac{k}{k+i\omega}e^{i\omega t} = \frac{1}{1+i(\omega/k)}\,e^{i\omega t}. \tag{19}$$

This is our complex solution; writing the coefficient in polar form as shown on the left below (you'll see it's better to use $-\phi$), we get the complex solution in the form on the right:

$$\frac{1}{1+i(\omega/k)} = Ae^{-i\phi}, \qquad \tilde{y} = A\,e^{i(\omega t-\phi)}. \tag{20}$$

The modulus A and argument $-\phi$ are calculated by the rules in Notes C:

$$A = \frac{1}{|1+i(\omega/k)|} = \frac{1}{\sqrt{1+(\omega/k)^2}}, \tag{21}$$

$$\phi = \arg(1+i(\omega/k)) = \tan^{-1}(\omega/k). \tag{22}$$

The real and imaginary parts of this complex response $\tilde{y} = A\,e^{i(\omega t-\phi)}$ give the responses of the system to respectively the real and imaginary parts of the complex input $e^{i\omega t}$; thus we can summarize our work as follows:

First-order Sinusoidal Input Theorem. *For the equation* $y' + ky = kq_e$ *we have*

$$\textit{physical input } q_e\text{:} \begin{cases} \cos\omega t \\ \sin\omega t \end{cases} \qquad \textit{response:} \begin{cases} A\cos(\omega t-\phi) \\ A\sin(\omega t-\phi) \end{cases}, \tag{23}$$

where the amplitude A and phase lag ϕ are given by (21) and (22).

6. Sinusoidal oscillations: reference.

The terms in (23) are part of the vocabulary describing a pure, or **sinusoidal** oscillation: one that can be written in the form

$$A\,\cos(\omega t - \phi). \tag{24}$$

$|A|$ is its **amplitude**: how high its graph rises over the t-axis at its maximum points;

ϕ is its **phase lag**: the value of ωt for which the graph is at its maximum (if $\phi = 0$, the graph has the position of $\cos\omega t$; if $\phi = \pi/2$, it has the position of $\sin\omega t$);

ϕ/ω is its **time delay, time lag**: how far to the right on the t-axis the graph of $\cos\omega t$ has been moved to make the graph of (24); (to see this, write $A\cos(\omega t-\phi) = A\cos(\omega(t-\phi/\omega))$;

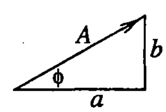
ω is its **angular frequency**: the number of complete oscillations it makes in a time interval of length 2π;

$\omega/2\pi$ (usually written ν) is its **frequency**: the number of complete oscillations the graph makes in a time interval of length 1;

$2\pi/\omega$ or $1/\nu$ is its **period**, the t-interval required for one complete oscillation.

The Sinusoidal Identity For any real constants a and b,

$$a\,\cos\theta + b\,\sin\theta = A\,\cos(\theta-\phi), \tag{25}$$



where A and ϕ can be described in at least three ways:

(26) $A = |\,a\,\mathbf{i} + b\,\mathbf{j}\,|$, $\quad\phi$ is the angle rotating $\mathbf{i}$ into $\mathrm{dir}(a\,\mathbf{i} + b\,\mathbf{j})$;

$$A = \sqrt{a^2+b^2}, \qquad \phi = \tan^{-1}\frac{b}{a}; \tag{27}$$

$$a + bi = A\,e^{i\phi}. \tag{28}$$