LC. Limit Cycles

1. Introduction.

In analyzing non-linear systems in the xy-plane, we have so far concentrated on finding the critical points and analysing how the trajectories of the system look in the neighborhood of each critical point. This gives some feeling for how the other trajectories can behave, at least those which pass near anough to critical points.

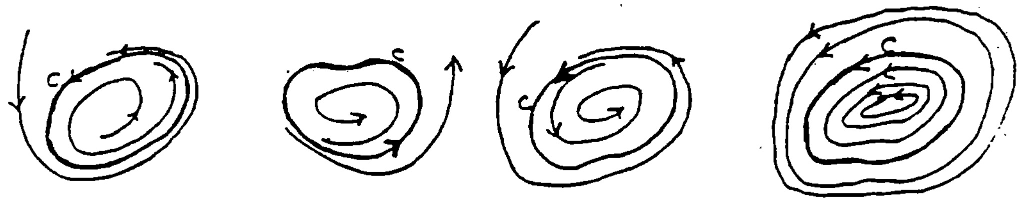
Another important possibility which can influence how the trajectories look is if one of the trajectories traces out a closed curve C. If this happens, the associated solution $\mathbf{x}(t)$ will be geometrically realized by a point which goes round and round the curve C with a certain period T. That is, the solution vector

$$\mathbf{x}(t) = (x(t), y(t))$$

will be a pair of periodic functions with period T :

$$x(t+T) = x(t), \qquad y(t+T) = y(t) \qquad \text{for all } t.$$

If there is such a closed curve, the nearby trajectories must behave something like C. The possibilities are illustrated below. The nearby trajectories can either spiral in toward C, they can spiral away from C, or they can themselves be closed curves. If the latter case does not hold — in other words, if C is an *isolated* closed curve — then C is called a *limit cycle*: stable, unstable, or semi-stable according to whether the nearby curves spiral towards C, away from C, or both.



STABLE LIMIT CYCLE — UNSTABLE LIMIT CYCLE — SEMI-STABLE LIMIT CYCLE — NEUTRALLY-STABLE CENTER

The most important kind of limit cycle is the stable limit cycle, where nearby curves spiral towards C on both sides. Periodic processes in nature can often be represented as stable limit cycles, so that great interest is attached to finding such trajectories if they exist. Unfortunately, surprisingly little is known about how to do this, or how to show that a system has no limit cycles. There is active research in this subject today. We will present a few of the things that are known.

2. Showing limit cycles exist.

The main tool which historically has been used to show that the system

$$x' = f(x,y)$$
$$y' = g(x,y) \tag{1}$$

has a stable limit cycle is the

Poincare-Bendixson Theorem *Suppose R is the finite region of the plane lying between two simple closed curves D_1 and D_2, and* **F***is the velocity vector field for the system (1). If*

(i) at each point of D_1 and D_2, the field **F***points toward the interior of R, and*

(ii) R contains no critical points,

then the system (1) has a closed trajectory lying inside R.

The hypotheses of the theorem are illustrated by fig. 1. We will not give the proof of the theorem, which requires a background in Mathematical Analysis. Fortunately, the theorem strongly appeals to intuition. If we start on one of the boundary curves, the solution will enter R, since the velocity vector points into the interior of R. As time goes on, the solution can never leave R, since as it approaches a boundary curve, trying to escape from R, the velocity vectors are always pointing inwards, forcing it to stay inside R. Since the solution can never leave R, the only thing it can do as $t \to \infty$ is either approach a critical point — but there are none, by hypothesis — or spiral in towards a closed trajectory. Thus there is a closed trajectory inside R. (It cannot be an unstable limit cycle—it must be one of the other three cases shown above.)

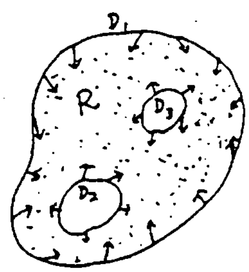


fig. 1.

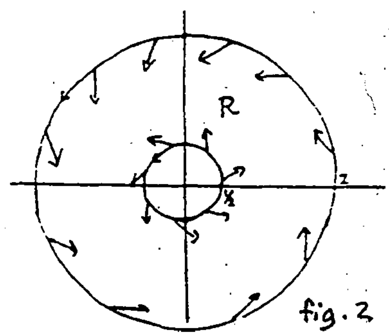


fig. 2

To use the Poincare-Bendixson theorem, one has to search the vector field for closed curves D along which the velocity vectors all point towards the same side. Here is an example where they can be found.

Example 1. Consider the system

$$x' = -y + x(1 - x^2 - y^2)$$
$$y' = x + y(1 - x^2 - y^2) \tag{2}$$

Figure 2 shows how the associated velocity vector field looks on two circles. On a circle of radius 2 centered at the origin, the vector field points inwards, while on a circle of radius 1/2, the vector field points outwards. To prove this, we write the vector field along a circle of radius r as

$$\mathbf{x}' = (-y\,\mathbf{i} + x\,\mathbf{j}) + (1 - r^2)(x\,\mathbf{i} + y\,\mathbf{j})\ . \tag{3}$$

The first vector on the right side of (3) is tangent to the circle; the second vector points radially *in* for the big circle ($r = 2$), and radially *out* for the small circle ($r = 1/2$). Thus the sum of the two vectors given in (3) points inwards along the big circle and outwards along the small one.

We would like to conclude that the Poincare-Bendixson theorem applies to the ring-shaped region between the two circles. However, for this we must verify that R contains no critical points of the system. We leave you to show as an exercise that $(0,0)$ is the only critical point of the system; this shows that the ring-shaped region contains no critical points.

The above argument shows that the Poincare-Bendixson theorem can be applied to R, and we conclude that R contains a closed trajectory. In fact, it is easy to verify that $x = \cos t,\ y = \sin t$ solves the system, so the unit circle is the locus of a closed trajectory. We leave as another exercise to show that it is actually a stable limit cycle for the system, and the only closed trajectory.

3. Non-existence of limit cycles

We turn our attention now to the negative side of the problem of showing limit cycles exist. Here are two theorems which can sometimes be used to show that a limit cycle does *not* exist.

Bendixson's Criterion *If f_x and g_y are continuous in a region R which is simply-connected (i.e., without holes), and*

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \neq 0 \quad \text{at any point of } R,$$

then the system

$$x' = f(x,y) \qquad (4)$$
$$y' = g(x,y)$$

has no closed trajectories inside R.

Proof. Assume there is a closed trajectory C inside R. We shall derive a contradiction, by applying Green's theorem, in its normal (flux) form. This theorem says

$$(5) \qquad \oint_C (f\,\mathbf{i} + g\,\mathbf{j}) \cdot \mathbf{n}\, ds \quad \equiv \quad \oint_C f\,dy - g\,dx \quad = \quad \iint_D \Big(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\Big)\,dx\,dy\ .$$

where D is the region inside the simple closed curve C.

This however is a contradiction. Namely, by hypothesis, the integrand on the right-hand side is continuous and never 0 in R; thus it is either always positive or always negative, and the right-hand side of (5) is therefore either positive or negative.

On the other hand, the left-hand side must be zero. For since C is a closed trajectory, C is always tangent to the velocity field $f\,\mathbf{i} + g\,\mathbf{j}$ defined by the system. This means the normal vector $\mathbf{n}$ to C is always perpendicular to the velocity field $f\,\mathbf{i} + g\,\mathbf{j}$, so that the integrand $f(f\,\mathbf{i} + g\,\mathbf{j}) \cdot \mathbf{n}$ on the left is identically zero.

This contradiction means that our assumption that R contained a closed trajectory of (4) was false, and Bendixson's Criterion is proved. □

Critical-point Criterion *A closed trajectory has a critical point in its interior.*

If we turn this statement around, we see that it is really a criterion for *non-existence*: it says that *if a region R is simply-connected (i.e., without holes) and has no critical points, then it cannot contain any limit cycles.* For if it did, the Critical-point Criterion says there would be a critical point inside the limit cycle, and this point would also lie in R since R has no holes.

(Note carefully the distinction between this theorem, which says that limit cycles enclose regions which *do* contain critical points, and the Poincare-Bendixson theorem, which seems to imply that limit cycles tend to lie in regions which *don't* contain critical points. The difference is that these latter regions always contain a hole; the critical points are in the hole. Example 1 illustrated this.

Example 2. For what a and d does $\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases}$ have closed trajectories?

Solution. By Bendixson's criterion, $a + d \neq 0 \quad \Rightarrow$ no closed trajectories.

What if $a + d = 0$? Bendixson's criterion says nothing. We go back to our analysis of the linear system in Notes LS. The characteristic equation of the system is

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0\ .$$

Assume $a + d = 0$. Then the characteristic roots have opposite sign if $ad - bc < 0$ and the system is a saddle; the roots are pure imaginary if $ad - bc > 0$ and the system is a center, which has closed trajectories. Thus

$$\textit{the system has closed trajectories} \quad \Leftrightarrow \quad a + d = 0, \quad ad - bc > 0.$$

4. The Van der Pol equation.

An important kind of second-order non-linear autonomous equation has the form

$$x'' + u(x)\,x' + v(x) = 0 \qquad \textit{(Liénard equation)}\ . \tag{6}$$

One might think of this as a model for a spring-mass system where the damping force $u(x)$ depends on position (for example, the mass might be moving through a viscous medium of varying density), and the spring constant $v(x)$ depends on how much the spring is stretched—this last is true of all springs, to some extent. We also allow for the possibility that $u(x) < 0$ (i.e., that there is "negative damping").

The system equivalent to (6) is

$$\begin{aligned} x' &= y \\ y' &= -v(x) - u(x)\,y \end{aligned} \tag{7}$$

Under certain conditions, the system (7) has a unique stable limit cycle, or what is the same thing, the equation (6) has a unique periodic solution; and all nearby solutions tend towards this periodic solution as $t \to \infty$. The conditions which guarantee this were given by Liénard, and generalized in the following theorem.

Levinson-Smith Theorem *Suppose the following conditions are satisfied.*

(a) $u(x)$ *is even and continuous,*

(b) $v(x)$ *is odd,* $v(x) > 0$ *if* $x > 0$, *and* $v(x)$ *is continuous for all* x,

(c) $V(x) \to \infty$ *as* $x \to \infty$, *where* $V(x) = \int_0^x v(t)\,dt$,

(d) *for some* $k > 0$, *we have*

$$\left.\begin{array}{ll} U(x) < 0, & \text{for } 0 < x < k, \\ U(x) > 0 \text{ and increasing}, & \text{for } x > k, \\ U(x) \to \infty, & \text{as } x \to \infty, \end{array}\right\} \qquad \text{where} \quad U(x) = \int_0^x u(t)\,dt.$$

Then, the system (7) has

i) a unique critical point at the origin;

ii) a unique non-zero closed trajectory C, *which is a stable limit cycle around the origin;*

iii) all other non-zero trajectories spiralling towards C *as* $t \to \infty$.

We omit the proof, as too difficult. A classic application is to the equation

$$x'' - a(1 - x^2)\,x' + x = 0 \qquad \textit{(van der Pol equation)} \tag{8}$$

which describes the current $x(t)$ in a certain type of vacuum tube. (The constant a is a positive parameter depending on the tube constants.) The equation has a unique non-zero periodic solution. Intuitively, think of it as modeling a non-linear spring-mass system. When $|x|$ is large, the restoring and damping forces are large, so that $|x|$ should decrease with time. But when $|x|$ gets small, the damping becomes negative, which should make $|x|$ tend to increase with time. Thus it is plausible that the solutions should oscillate; that it has exactly one periodic solution is a more subtle fact.

There is a lot of interest in limit cycles, because of their appearance in systems which model processes exhibiting periodicity. Not a great deal is known about them.

For instance, it is not known how many limit cycles the system (1) can have when $f(x, y)$ and $g(x, y)$ are quadratic polynomials. In the mid-20th century, two well-known Russian mathematicians published a hundred-page proof that the maximum number was three, but a gap was discovered in their difficult argument, leaving the result in doubt; twenty years later the Chinese mathematician Mingsu Wang constructed a system with four limit cycles. The two quadratic polynomials she used contain both very large and very small coefficients; this makes numerical computation difficult, so there is no computer drawing of the trajectories.

There the matter currently rests. Some mathematicians conjecture the maximum number of limit cycles is four, others six, others conjecture that there is no maximum. For autonomous systems where the right side has polynomials of degree higher than two, even less is known. There is however a generally accepted proof that for any particular system for which $f(x, y)$ and $g(x, y)$ are polynomials, the number of limit cycles is finite.

Exercises: Section 5D