Recitation 1, February 7, 2006

Natural growth and decay

Topics for discussion:

1. Natural growth and decay as an example of a model leading to a differential equation

2. Separable equations. Here it is important to understand what function has derivative $1/x$; this is $\ln|x| + c$, not just $\ln x + c$.

Natural growth and decay

Think about what happens in the process of radioactive decay between time t and time $t + \Delta t$. If we get a variable x in place, we are led to

$$x(t + \Delta t) \simeq x(t) - k\, x(t)\, \Delta t.$$

Each of the three factors k, x, Δt have a meaning. Then $\dot{x} = -kx$. Not quite the same form as $\dot{x} = f(t)$, which is the subject of integral calculus: it seems you have to know x already. T solve it: cross multiply to get $dx/x = -k\,dt$ and then integrate both sides. Amalgamate the constants: $\ln x = -kt + c$. Exponentiate: $x = e^c e^{-kt}$. Draw some solution curves. Point out that if $x < 0$ then $\frac{d}{dx}\ln(-x) = \frac{1}{x}$, so it's better to write $\ln|x|$ and then trade the absolute value for $\pm e^c$. There's also the "missing solution" $x = 0$, so in the end $x = Ce^{-kt}$. You should memorize the fact that $\dot{x} = kx$ has general solution $x = Ce^{-kt}$. Note $C = x(0)$. A lesson: differential equations have many solutions, indexed by a "constant of integration."

Class work on a project

Some biologists are doing research on an ant population. They mark out a 2-meter by 2-meter square and study the ants in that square. They don't control the migration of ants in or out of the square. Suppose (an unnatural assumption, but we'll make it anyway) that there is a net emigration of ants out of the square with a rate of a ants per day. Suppose ants reproduce according to natural growth, at k new ants per ant per day.

1. Write down a model for the ant population.

2. Find the general solution of this equation. [This will involve a review of simple change of variables in an integration, something the students should be comfortable with but often are not.]

3. Check that the proposed solution satisfies the ODE.

4. There is a "steady state" (also known as constant, or equilibrium) solution. Find it. Does the way the solution depends upon k and a make sense? (That is: do the units come out right? Does it behave right when a is large, or small, and when k is large, or small?)