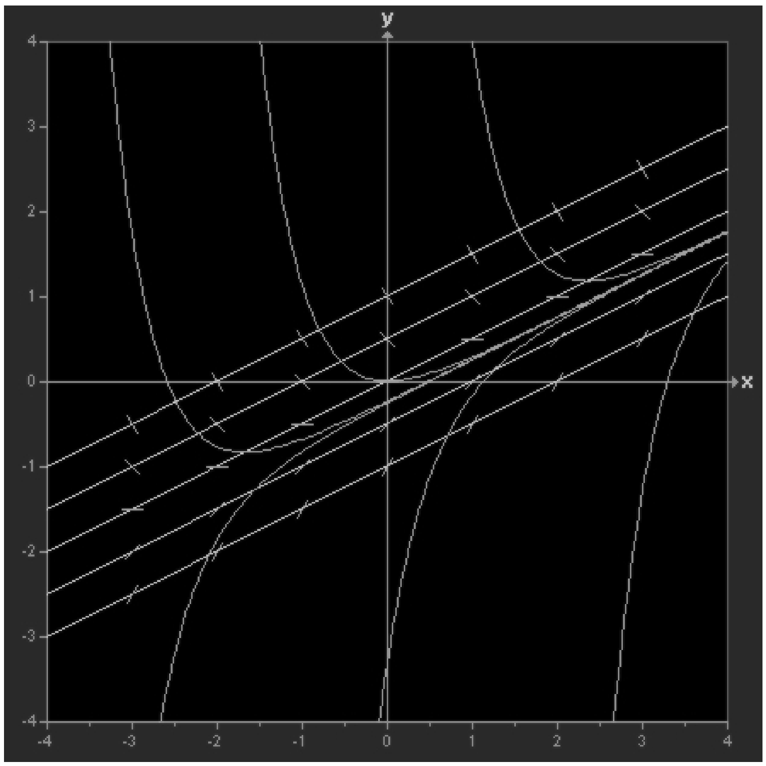
Recitation 2, February 9, 2006

Direction fields, integral curves, isoclines

Solution suggestions

1. Draw a big axis system and plot some isoclines, especially the nullcline. Plot a few solutions.

Here is a picture of some isoclines and a few solutions created by the Mathlet 'Isoclines'.



2. One of the integral curves seems to be a straight line. Is this true? What straight line is it? (i.e. for what m and b is $y = mx + b$ a solution?)

We want the straight line $y = mx + b$ to be an integral curve. We compute

$$y' = m$$

and

$$x - 2y = (1 - 2m)x - 2b\ .$$

For the two to be equal we must have $m = \frac{1}{2}$ and then $b = -\frac{1}{4}$. The straight line that is a solution is then

$$y = \frac{1}{2}x - \frac{1}{4}\ .$$

3. As a general thing, if a straight line is an integral curve, how is it related to the isoclines of the equation? What happens in our example?

If a straight line is a solution, then the direction field along that straight line has to be constant, with slope equal to the slope of the line; so the solution is an isocline.

In our case $F(x, y) = x - 2y$, so the isocline $F(x, y) = c$ can be written as the equation for a straight line,

$$y = \frac{1}{2}x - \frac{c}{2} .$$

Since the slope has to be equal to the directional field we obtain $c = \frac{1}{2}$, so it checks.

4. It seems that all the solutions become asymptotic as $x \to \infty$. Explain at least why solutions get trapped between parallel lines of some fixed slope .

If an integral curve crosses the "null-cline" it has to stay underneath of it for increasing x. The reason is that the slope of the isocline is $\frac{1}{2}$, and thus steeper than the value of the directional field along this isocline which is 0. We also determined that the isocline for $c = \frac{1}{2}$ is an integral curve. But integral curves cannot cross each other. Thus the solution has to stay between the isocline for $c = 0$ and $c = \frac{1}{2}$. One can repeat this argument replacing the upper isocline, i.e. the null-cline, by the isocline for $c = \frac{1}{4}, \frac{3}{8}, \ldots$ and so on. We then see that the asymptotics is in fact given by the isocline for $c = \frac{1}{2}$.

For integral curves coming from below, we can make the same argument by looking at the isoclines for $c = 1$ and $c = \frac{1}{2}$.

5. What can be said in general about when a solution has a critical point? Where are the critical points of the solutions in our example? How many critical points can a single solution have? Can you predict on the basis of an initial value whether or not a solution will have a critical point? When there is one, is it a minimum or a maximum?

A critical point is given by $y' = 0$. Our differential equation is $y' = F(x, y)$. Therefore, we know that the critical points must lie on the null-cline. In general, integral curves can have more than one critical point.

In the case of $F(x, y) = x - 2y$ however, an integral curve can intersect the nullcline at most one time (as explained in the answer to question 4). So it can have at most one critical point. The question is whether an integral curve in fact crosses this isocline at all. The xy-plane is divided into two halfs by the isocline for $c = \frac{1}{2}$ which is also an integral curve. No integral curve can cross this line. A solution that starts below this line will never be able to reach the isocline for $c = 0$ above, thus can never have a critical point.

What about the solutions that start above the isocline for $c = \frac{1}{2}$? If you start with an integral curve that starts above the isocline for $c = -1$, say, you see it is crossing the null-cline and is then asymptotically approaching the isocline for $c = \frac{1}{2}$. Thus, this solution has a critical point. You will see that starting above the isocline for $c = \frac{1}{2}$ is really enough – it means that for some x (which might be far on the left side) it has crossed the null-cline. Now, is this critical point a maximum or minimum? We take the derivative of our differential equation and obtain

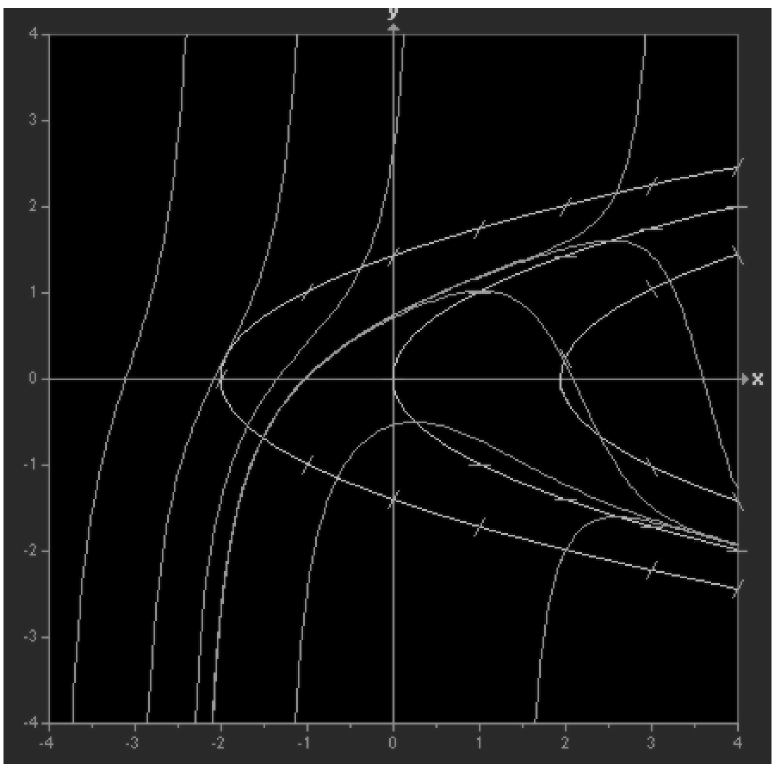
$$y'' = \left(x - 2y\right)' = 1 - 2y' \ .$$

Along the null-cline $y' = 0$. Therefore, $y'' = 1$ along the null-cline. The critical point is a minimum.

Finally, the integral curve $c = \frac{1}{2}$ itself does not have a critical point.

6. In lecture the equation $y' = y^2 - x$ was discussed. There is more to say about that example than there was time to describe. Sketch some isoclines and some solutions. One question is: where are the critical points of solutions? Can a solution have more than one?

Here is a picture of some isoclines and a few solutions created by the Mathlet 'Isoclines'.



Critical points are found if the solution crosses the null-cline, i.e. the parabola $y^2 = x$ (opening to the right). Once a solution crosses the upper branch of this isocline it cannot cross the lower branch as well (which would then make it having two critical points). The reason is the following: the lower branch of the parabola is described by $y = -\sqrt{x}$. So the *slope of this isocline itself* is $-\frac{1}{2\sqrt{x}}$ which is always negative (pointing downwards). The value of the directional field on the null-cline is zero and thus steeper (or pointing horizontally). Thus, a solution cannot cross the lower branch of the null-cline.

7. How about points of inflection (where $y'' = 0$)? Hint: differentiate the ODE and then replace y' with the right hand side of the original ODE. (You may want to think about what happens in the $y' = x - 2y$ example as well.)

We have

$$y'' = \left(y^2 - x\right)' = 2yy' - 1 = 2y(y^2 - x) - 1 \ .$$

Setting the RHS equal to zero we see that the points of inflection occur along the graph of

$$x = y^2 - \frac{1}{2y}$$

in the xy-plane.

In the example $y' = x - 2y$ we obtain

$$y'' = 1 - 2y' = 1 - 2x + 4y \ .$$

Setting the RHS equal to zero, we obtain the equation of the line

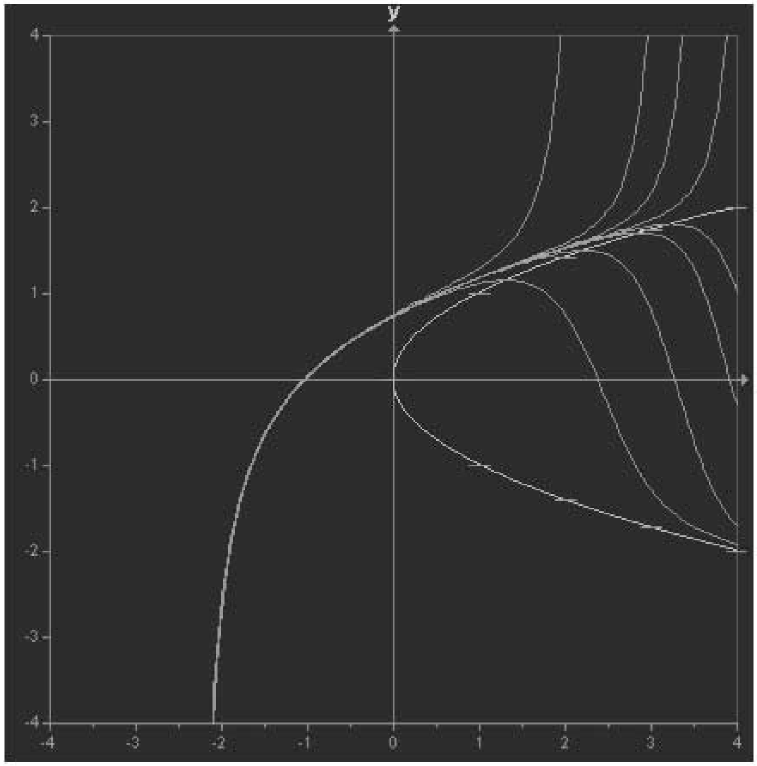
$$y = \frac{1}{2}x - \frac{1}{4} \ .$$

The point is that every point along a straight line is a point of inflection, according to the definition $y" = 0$. The points of the straight line integral curve are the only points of inflection on any integral curve.

8. A "separatrix" is a solution such that solutions on one side of it have a fate entirely different from solutions on the other side. The equation $y' = y^2 - x$ exhibits a separatrix. Sketch it and describe the differing behaviors.

If you play with the Mathlet you will see that there are two types of solutions. Either a solution $y(x)$ blows up in a finite time x or the solution approaches the lower branch of $y^2 = x$ which means $y(x) \to -\sqrt{x}$ for $x \to \infty$.

Here is a picture of solutions near the separatrix created by the Mathlet 'Isoclines'.



y
4
3
2
1
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-4
-3
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2
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4