Recitation 4, February 16, 2006

First order Linear ODEs: Models and Solutions

$$\dot{x} + p(t)x = q(t)$$

Variation of parameters: Find a nonzero solution of $\dot{x}_h + p(t)x_h = 0$. Then write $x = ux_h$ and solve for u.

Formulas:

$$x_h = e^{-\int p(t)\,dt}, \qquad x = x_h \int x_h^{-1} q\,dt + cx_h$$

1. Do EP 1.4: 38: A cascade of tanks, of volumes 100 gal and 200 gal, each initially with 50 lb of salt; flow rate is 5 gal/minute, with pure water coming into the top tank. Write $x(t)$ for the number of pounds of salt in the top tank, $y(t)$ for the number of pounds of salt in the second.

(a) Come up with a differential equation for $x(t)$ and solve it.

(b) Explain why $\dfrac{dy}{dt} = \dfrac{5x}{100} - \dfrac{5y}{200}$.

(c) Then solve for $y(t)$.

(d) What is the maximum amount of salt ever in the second tank?

2. Find the general solution of $\dot{x}+2x = 4t$. Plot the direction field and some of the solutions. What is the particular solution with $x(0) = 0$? Do all solutions converge as $t \to \infty$?

3. Recognize the left hand side as the derivative of a product in order to find the general solution of $x^2y' + 2xy = \sin(2x)$.

4. Suppose x_h is a nonzero solution to $\dot{x} + p(t)x = 0$. Show that

$$\frac{d}{dt}(x_h^{-1}x) = x_h^{-1}(\dot{x} + p(t)x)$$

This shows that x_h^{-1} is an "integrating factor": when you multiply $\dot{x}+p(t)x = q(t)$ by it, the left hand side becomes the derivative of the product $x_h^{-1}x$.

Use this method to solve $\dot{x} + 3x = 9t$ again.