Recitation 4, February 16, 2006

First order Linear ODEs: Models and Solutions

Solution suggestions

1. Do EP 1.4: 38: A cascade of tanks, of volumes 100 gal and 200 gal, each initially with 50 lb of salt; flow rate is 5 gal/minute, with pure water coming into the top tank. Write $x(t)$ for the number of pounds of salt in the top tank, $y(t)$ for the number of pounds of salt in the second.

(a) Come up with a differential equation for $x(t)$ and solve it.

We have to determine how the amount of $x(t)$ lb of salt in the first tank changes over a small time interval of Δt minutes. First, since pure water is running into the first tank we are not adding any salt. Second, we have to determine how much salt is flowing out. Remember that we are still assuming that at all times the solution in the tanks is kept uniformly mixed. That means that at time t minutes the concentration of salt in the tank is $x(t)/V_1$ lb/gal. The *amount* flowing out is thus $\frac{x(t)}{V_1}\, r\,\Delta t$ with r the flow rate of 5 gal/minute and V_1 the volume of tank 1. We obtain

$$\Delta x \simeq -\frac{x}{V_1}\, r\,\Delta t\ .$$

The units on the RHS combine to lb. Dividing by Δt and taking the limit we obtain the differential equation

$$\dot{x} = -\frac{x}{V_1}\, r = -\frac{5}{100}x = -\frac{1}{20}x\ .$$

The initial condition is $x(0) = 50$. This is the same ODE as for the radioactive decay. The solution is $x(t) = C_1\, e^{-\frac{1}{20}t}$. The initial condition gives $C_1 = 50$. Thus the amount of salt in tank 1 is described by the function

$$x(t) = 50\, e^{-\frac{1}{20}t}\ .$$

(b) Explain why $\dfrac{dy}{dt} = \dfrac{5x}{100} - \dfrac{5y}{200}$.

Again, we have to determine how the amount of $y(t)$ lb of salt in tank 2 changes over a small time interval Δt minutes. First, we determine the amount of salt that we are adding over Δt minutes. The saline solution that is flowing in has the concentration $x(t)/V_1$ as it is coming from tank 1. This adds the amount of $\frac{x(t)}{V_1}\, r\,\Delta t$ lb of salt. The units of $x(t)/V_1, r$, and Δt combine to give lb. Second, we have to determine how much salt is flowing out. Remember that we are assuming that at all times the solution in the tank 2 is kept uniformly mixed as well. That means that at time t minutes the concentration of salt in the tank 2 is $y(t)/V_2$ lb/liter with V_2 the volume of tank 2. The *amount* flowing out is thus $\frac{y(t)}{V_2}\, r\,\Delta t$. Again, the units combine to lb. Combining these contributions we obtain

$$\Delta y \simeq \frac{x}{V_1}\, r\,\Delta t - \frac{y}{V_2}\, r\,\Delta t\ .$$

Dividing by Δt and taking the limit we obtain the differential equation

$$\dot{y} = \frac{r\,x(t)}{V_1} - \frac{r\,y(t)}{V_2}\ .$$

with the values for $V_1 = 100, V_2 = 200, r = 5$ this is $\dfrac{dy}{dt} = \dfrac{5x}{100} - \dfrac{5y}{200}$. Written in the standard form this ODE becomes

$$\dot{y} + \frac{5}{200}\,y = \frac{5}{100}\,50e^{-\frac{1}{20}t}$$

or

$$\dot{y} + \frac{1}{40}\,y = \frac{5}{2}e^{-\frac{1}{20}t}\ .$$

The initial condition is $y(0) = 50$.

(c) Then solve for $y(t)$.

We have $p(t) = \frac{1}{40}$ and $q(t) = \frac{5}{2}e^{-\frac{1}{20}t}$. We find

$$y_h(t) = e^{-\int \frac{1}{40}dt} = e^{-\frac{t}{40}}\ .$$

With $y = y_h \int y_h^{-1} q\,dt + C y_h$ we obtain

$$y = e^{-\frac{t}{40}} \int e^{\frac{t}{40}}\,\frac{5}{2}e^{-\frac{1}{20}t}dt + Ce^{-\frac{t}{40}}\ .$$

This is

$$y = \frac{5}{2}\,e^{-\frac{t}{40}} \int e^{-\frac{t}{40}}dt + Ce^{-\frac{t}{40}}\ .$$

Computing the integral yields

$$y = -100\,e^{-\frac{t}{20}} + Ce^{-\frac{t}{40}}\ .$$

Our initial condition is $y(0) = 50$. This determines $C = 150$ and thus

$$y = -100\,e^{-\frac{t}{20}} + 150e^{-\frac{t}{40}}\ .$$

(d) What is the maximum amount of salt ever in the second tank?

We want to find T such that $\dot{y}(T) = 0$. Taking the derivative and setting it equal to zero we obtain

$$5e^{-\frac{T}{20}} - \frac{15}{4}e^{-\frac{T}{40}} = 0\ .$$

Let us call $s = e^{-\frac{T}{40}}$. Then the LHS becoms

$$5s^2 - \frac{15}{4}s = 0\ .$$

Therefore, $s = 0$ or $s = \frac{3}{4}$. If $s = \frac{3}{4}$ then the amount of salt $y(T)$ is

$$y(T) = -100\,e^{-\frac{T}{20}} + 150e^{-\frac{T}{40}} = -100\,s^2 + 150s = 56.25\ .$$

$s = 0$ corresponds to the limit of very large T. We see from the solution y that for t very large $y(t)$ goes to zero. We also know that at $t = 0$ we have $y(0) = 50$ from our initial condition. Since there is only one critical point for $y(t)$ for finite t with value $y(t) = 56.25$ which is bigger than both the initial amount ($t = 0$) and the limiting amount (t very big), 56.25 lb is the maximum amount of salt ever in the second tank.

2. Find the general solution of $\dot{x}+2x = 4t$. Plot the direction field and some of the solutions. What is the particular solution with $x(0) = 0$? Do all solutions converge as $t \to \infty$?

The integrating factor is $x_h = e^{-2t}$. The general solution is

$$x = e^{-2t} \int 4t\, e^{2t}\, dt + Ce^{-2t} \, .$$

Integration by parts gives

$$\int 4t\, e^{2t} = 4t\, \frac{e^{2t}}{2} - \int 4\, \frac{e^{2t}}{2}\, dt = 2t\, e^{2t} - e^{2t} \, .$$

Thus, the general solution is

$$x = 2t - 1 + Ce^{-2t} \, .$$

The solution with $x(0) = 0$ has $C = 1$. For t very large the exponential term becomes very small and we have

$$x \simeq 2t - 1 \, .$$

No solution converges for $t \to \infty$.

3. Recognize the left hand side as the derivative of a product in order to find the general solution of $x^2 y' + 2xy = \sin(2x)$.

We see that $(x^2 y)' = x^2\, y' + 2x\, y$ by the product rule. Let us call $x^2 y$ a function f. Then the ode becomes

$$f' = \sin(2x) \, .$$

We integrate to obtain $f(x) = -\frac{1}{2}\cos(2x) + C$. Replacing f by $x^2\, y$ and dividing by x^2 we get

$$y(x) = -\frac{1}{2x^2}\cos(2x) + \frac{C}{x^2}$$

for the general solution.

4. Suppose x_h is a nonzero solution to $\dot{x} + p(t)x = 0$. Show that

$$\frac{d}{dt}(x_h^{-1} x) = x_h^{-1}(\dot{x} + p(t)x)$$

This shows that x_h^{-1} is an "integrating factor": when you multiply $\dot{x}+p(t)x = q(t)$ by it, the left hand side becomes the derivative of the product $x_h^{-1} x$.

Use this method to solve $\dot{x} + 3x = 9t$ again.

Remember that for a general function $f(t)$ we have

$$\frac{d}{dt}\Big(\frac{1}{f(t)}\Big) = -\frac{\dot{f}(t)}{f(t)^2} \ .$$

We have

$$\frac{d}{dt}(x_h^{-1}x) = \frac{d}{dt}(x_h^{-1})\, x + x_h^{-1}\, \frac{d}{dt}(x) \ ,$$

and thus

$$\frac{d}{dt}(x_h^{-1}x) = -\frac{\dot{x}_h}{x_h^2}\, x + x_h^{-1}\, \dot{x} \ .$$

Or we could have written

$$\frac{d}{dt}(x_h^{-1}x) = x_h^{-1}\Big(\dot{x} - \dot{x}_h\, x_h^{-1}\, x\Big) \ . \tag{1}$$

So far, we have been using just the general rules on how to take derivatives. But remember that x_h is not any function. It's a solution to

$$\dot{x}_h + p(t)x_h = 0 \ .$$

Or we could write $\dot{x}_h = -p(t)x_h$. Plugging this into the RHS of (1) we get

$$\begin{aligned} \frac{d}{dt}(x_h^{-1}x) &= x_h^{-1}\Big(\dot{x} - (-p(t)x_h)\, x_h^{-1}\, x\Big) \\ &= x_h^{-1}\Big(\dot{x} + p(t)\, x\Big) \ . \end{aligned}$$

In the example $\dot{x} + 3x = 9t$ we have $x_h = e^{-\int 3dt} = e^{-3t}$. Thus, the integrating factor is $x_h^{-1} = e^{3t}$. We multiply both sides of the ODE with this integrating factor and obtain

$$e^{3t}\,\dot{x} + 3e^{3t}\, x = 9t\, e^{3t} \ .$$

We realize that the LHS is just $\frac{d}{dt}(e^{3t}x)$. Therefore, we obtain

$$e^{3t}x = \int 9t\, e^{3t}\, dt + C \ . \tag{2}$$

Integration by parts gives us

$$\int 9t\, e^{3t} = 9t\,\frac{e^{3t}}{3} - \int 9\,\frac{e^{3t}}{3}\, dt = 3t\, e^{3t} - e^{3t} \ .$$

Eq. (2) becomes

$$e^{3t}x = 3t\, e^{3t} - e^{3t} + C$$

or

$$x = 3t - 1 + Ce^{-3t} \ .$$