Recitation 5, February 23, 2006

Complex numbers, complex exponentials

Solution suggestions

1. Mark the points $e^{i\frac{\pi}{3}}$ and $e^{\ln 2+i\frac{\pi}{4}}$ complex plane and find their expressions in the form $a+bi$.

$e^{i\frac{\pi}{3}}$ is the point on the unit circle (about the origin) which you obtain when you start at its intersection with the positive x-axis and move counterclockwise for an angle of 60 degrees (the one-o'clock-position). We write by the Euler-formula

$$e^{i\frac{\pi}{3}} = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2}\ .$$

$e^{\ln 2+i\frac{\pi}{4}} = 2e^{i\frac{\pi}{4}}$ is the point on the circle of radius 2 (about the origin) which you obtain when you start at its intesection with the positive x-axis and move counterclockwise for an angle of 45 degrees. We write by the Euler-formula

$$e^{\ln 2+i\frac{\pi}{4}} = 2e^{i\frac{\pi}{4}} = 2\Big(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\Big) = \sqrt{2} + i\sqrt{2}\ .$$

2. Express the fourth roots of -1 in the form $a+bi$.

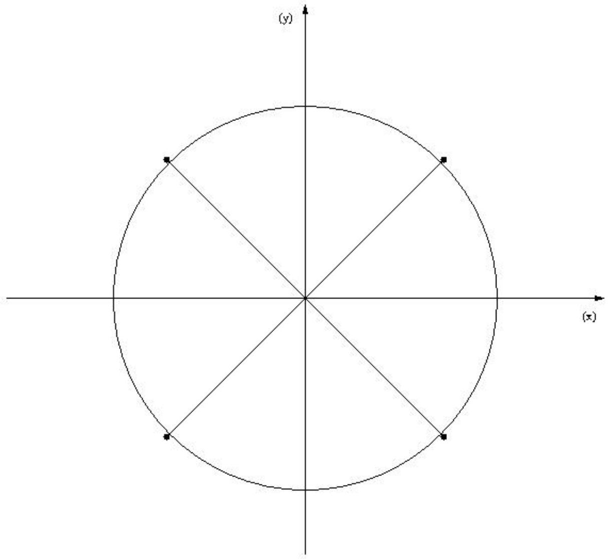
The fourth roots of unity are $e^{i\frac{2\pi}{4}k} = e^{i\frac{\pi}{2}k}$ where k is $0,1,2,3$. We have to multiply these four roots of unity with the principal value for the fourth root of $-1 = e^{i\pi}$ which is $e^{i\frac{\pi}{4}}$. Therefore, we obtain for the fourth roots of -1 the complex numbers $e^{i(\frac{\pi}{2}k+\frac{\pi}{4})}$ for $k=0,1,2,3$ or

$$e^{i\frac{\pi}{4}}\ ,\quad e^{i\frac{3\pi}{4}}\ ,\quad e^{i\frac{5\pi}{4}}\ ,\quad e^{i\frac{7\pi}{4}}\ .$$

$e^{i\frac{\pi}{4}}$ is the point on the unit circle which you obtain when you start at its intesection with the positive x-axis and move counterclockwise for an angle of 45 degrees. We obtain in the same way as in Problem 1

$$e^{i\frac{\pi}{4}} = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\ .$$

You can now already draw a picture of all the roots in the complex plane. Notice that you always move counterclockwise for an angle of 90 degrees, i.e. $\frac{\pi}{2}$, on the unite circle to get from one root to the next. You will get the following picture:



From the picture you see

$$
\begin{aligned}
e^{i\frac{3\pi}{4}} &= -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\,, \\
e^{i\frac{5\pi}{4}} &= -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\,, \\
e^{i\frac{7\pi}{4}} &= \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\,.
\end{aligned}
$$

3. Find the general solution of

$$\dot{x} + 2x = e^t\,. \tag{1}$$

Remember, to find a solution of

$$\dot{x} + p(t)x = q(t)$$

you have to find a nonzero solution of $\dot{x}_h + p(t)x_h = 0$ first. Then write $x = ux_h$ and solve for u. The formulas are:

$$x_h = e^{-\int p(t)\,dt}\,, \qquad x = x_h \int x_h^{-1} q\,dt + cx_h$$

In our example we have $x_h = e^{-2t}$ and

$$x = e^{-2t} \int e^{3t} dt + ce^{-2t} = \frac{1}{3}e^t + ce^{-2t}\,. \tag{2}$$

In the lecture, you have learned that you can also obtain a particular solution by plugging Ae^t into the differential equation (1) and solving for A. The idea is that a particular solution has the same form as the input function on the RHS of Eq. (1). Plugging Ae^t into the Eq. (1) we obtain on the LHS

$$\dot{x} + 2x = 3Ae^t \ .$$

The RHS of Eq. (1) is equal to e^t, so we find $A = \frac{1}{3}$. Thus, a particular solution to the differential equation is given by $\frac{1}{3}e^t$. From this, the general solution is obtained by adding the solution of the homogeneous ODE

$$\dot{x} + 2x = 0 \ .$$

The solution to the homogeneous ODE is ce^{-2t}. We obtain for the general solution of (1)

$$x = \frac{1}{3}e^t + ce^{-2t} \ , \tag{3}$$

which agrees with (2).

4. Solve $\dot{x} + 2x = \cos(2t)$ by replacing it with a complex valued equation, solving that, and then extracting the real part.

We write the complex equation

$$\dot{z} + 2z = e^{2it} \ .$$

Once we have found a solution $z(t)$, then a solution to our original ODE will be given by $x(t) = \text{Re}z(t)$.

We repeat the steps from Problem 3 and find $z_h = e^{-2t}$ and

$$z = e^{-2t} \int e^{(2+2i)t} dt + ce^{-2t} = \frac{1}{2+2i} e^{2it} + ce^{-2t} \ .$$

We now have to find the real part of $z(t)$: notice that

$$\frac{1}{2+2i} = \frac{2-2i}{8} = \frac{1-i}{4} \ .$$

Therefore, we have (remember $i^2 = -1$)

$$\text{Re}\Big(\frac{1-i}{4} e^{2it}\Big) = \text{Re}\Big(\frac{1-i}{4}(\cos(2t) + i\sin(2t))\Big) = \frac{1}{4}\Big(\cos(2t) + \sin(2t)\Big) \ .$$

Finally, we obtain

$$x(t) = \frac{1}{4}\Big(\cos(2t) + \sin(2t)\Big) + ce^{-2t} \ .$$

5. Write each of the following functions $f(t)$ in the form $A\cos(\omega t - \phi)$. In each case, begin by drawing a right triangle with sides a and b. (a) $\cos(2t) + \sin(2t)$. (b) $\cos(\pi t) - \sqrt{3}\sin(\pi t)$. (c) $\text{Im}\, \frac{e^{it}}{2+2i}$.

Taking the real part of the equation

$$e^{i\omega t}e^{-i\phi} = e^{i(\omega t-\phi)}$$

we obtain the *angle difference formula*

$$\cos(\omega t)\cos(\phi) + \sin(\omega t)\sin(\phi) = \cos(\omega t - \phi)\ .$$

Multiplying with A we obtain

$$\cos(\omega t)\ \underbrace{A\cos(\phi)}_{=a} + \sin(\omega t)\ \underbrace{A\sin(\phi)}_{=b} = A\cos(\omega t - \phi)\ .$$

Thus, our goal is to determine ϕ and A when a and b are given. If we draw a right triangle with sides a and b, then the length of the hypotenuse is A and the angle between A and the positive x-axis is ϕ.

(a) $a = b = 1$, thus $A = \sqrt{2}$ and $\phi = \frac{\pi}{4}$ (45 degrees), thus

$$\cos(2t) + \sin(2t) = \sqrt{2}\cos\left(2t - \frac{\pi}{4}\right)\ .$$

(b) $a = 1$, $b = -\sqrt{3}$. Then $A = 2$ and $\phi = -\frac{\pi}{3}$ (-60 degrees), thus

$$\cos(\pi t) - \sqrt{3}\sin(\pi t) = 2\cos\left(\pi t + \frac{\pi}{3}\right)\ .$$

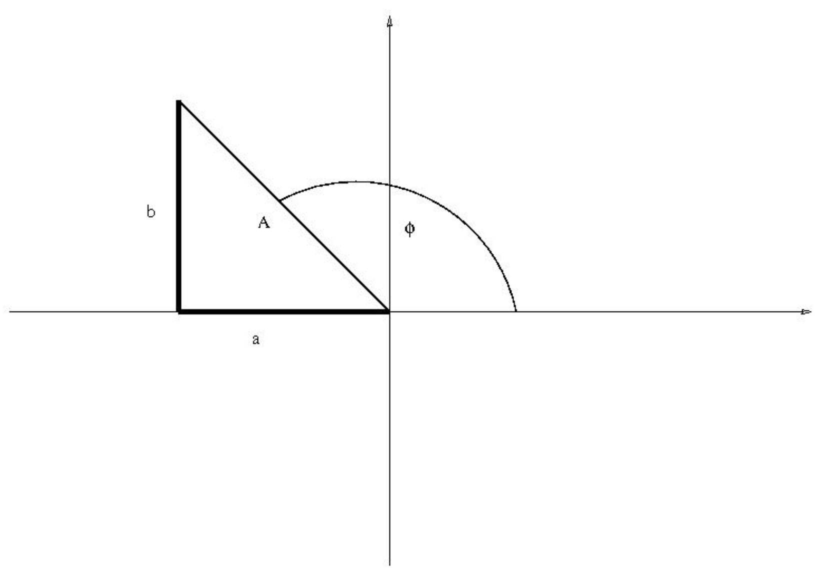
(c) We have

$$\frac{1}{2+2i} = \frac{2-2i}{8} = \frac{1-i}{4}\ .$$

Therefore, we are dealing with the expression

$$\mathrm{Im}\,\frac{e^{it}}{2+2i} = -\frac{1}{4}\cos(t) + \frac{1}{4}\sin(t)\ .$$

We have $a = -\frac{1}{4}$, $b = \frac{1}{4}$. From a picture like the following



we see $A = \frac{\sqrt{2}}{4}$, and $\phi = \frac{3\pi}{4}$ (135 degrees). thus

$$-\frac{1}{4}\cos(t) + \frac{1}{4}\sin(t) = \frac{\sqrt{2}}{4}\cos\left(t - \frac{3\pi}{4}\right) .$$