Recitation 7, March 2, 2006

Solutions to second order ODEs

Solution suggestions

1. What does the inequality $f''(a) > 0$ mean about the graph of the function $f(x)$ at $x = a$? How about the inequality $f''(a) < 0$?

What does the equality $f''(x) = 0$ say about the function $f(x)$? That is: what are the solutions $y = f(x)$ of the equation $y'' = 0$?

Which of these functions satisfy $y(0) = 0$? Does the "initial value" determine the solution of a second order ODE?

$f''(a) > 0$ means that the slope of the first derivative $f'(x)$ is positive or increasing at $x = a$. This means the graph is *concave up* at $x = a$. On the other hand, $f''(a) < 0$ means that the slope of the first derivative $f'(x)$ is negative or decreasing at $x = a$. This means the graph is *concave down* at $x = a$.

$f''(a) = 0$ means that the funct ion has to be concave up and concave down at the same time. Thus, if the function $f(x)$ satisfies $f''(x) = 0$ for all x it must be a straight line. A general straight line can be written as $f(x) = mx + b$.

Let's try to obtain the same result by integrating the equation $f''(x) = 0$. A first integration gives

$$\int f''(x)\,dx = c_1$$

or $f'(x) = c_1$. Another integration leads to

$$\int f'(x)\,dx = \int c_1\,dx = c_1\,x + c_2\ .$$

If we set $c_1 = m$ and $c_2 = b$ we get the general straight line.

The solution $y(x) = mx+b$ satisfies $y(0) = 0$ exactly if $b = 0$. So all $y(x) = mx$ where we can pick m arbitrarily satisfy the differential equation $y'' = 0$ and the "initial condition" $y(0) = 0$. As we can still pick m the solution is not determined uniquely.

2. Find all the solutions of $y'' = x^3$. Suppose I add the "initial condition" $y(0) = 1$, $y'(0) = 2$?

By a first integration of the equation $y'' = x^3$ we obtain

$$\int y''(x)\,dx = \int x^3\,dx = \frac{1}{4}x^4 + c_1$$

or $y'(x) = \frac{1}{4}x^4 + c_1$. A second integration gives

$$\int y'(x)\,dx = \int (\frac{1}{4}x^4 + c_1)\,dx = \frac{1}{20}x^5 + c_1 x + c_2$$

or $y(x) = \frac{1}{20}x^5 + c_1 x + c_2$. The inbitial condition $y(0) = 1$ gives $c_2 = 1$. The initial condition $y'(0) = 2$ gives $c_1 = 2$. Thus, we have $y(x) = \frac{1}{20}x^5 + 2x + 1$ as the unique solution of $y''(x) = x^3$ that satisfies at the same time $y(0) = 1$ and $y'(0) = 2$.

3. Find all functions of the form $x = at^2 + bt + c$ which are solutions of $\ddot{x} + \dot{x} + 2x = 4t^2$.

For $x = at^2 + bt + c$ we compute $\dot{x} = 2at + b$ and $\ddot{x} = 2a$. Thus, we obtain

$$\ddot{x} + \dot{x} + 2x = 2a + (2at + b) + 2(at^2 + bt + c) = 2a\,t^2 + (2a + 2b)\,t + (2a + b + 2c) \ .$$

Since the RHS has to be equal to $4t^2$ we get the *three* equations by matching the coefficients

$$2a = 4 \ , \qquad 2a + 2b = 0 \ , \qquad 2a + b + 2c = 0 \ .$$

Starting at the equation from the highest degree we obtain $a = 2$, then $b = -2$, and then $c = -1$. Thus, $x(t) = 2t^2 - 2t - 1$.

4. Show that $\sin(5t)$ and $\cos(5t)$ are both solutions of the "harmonic oscillator $\ddot{x} + 25x = 0$.

Can you think of solutions to $\ddot{x} - 25x = 0$?

Since $\frac{d}{dt}\sin(5t) = 5\cos(5t)$ and $\frac{d}{dt}\cos(5t) = -5\sin(5t)$ we have $\ddot{x} = -25\,x$ for both $x(t) = \cos(5t)$ and $x(t) = \sin(5t)$.

Accordingly, for the equation $\ddot{x} - 25x = 0$ we can write $\ddot{x} = 25\,x$. As the exponential function e^{at} satisfies $\ddot{x} = a^2\,x$ we must have $a^2 = 25$ or $a = \pm 5$. This means that we have found the two solutions e^{5t} and e^{-5t}.

The connection between these two differential equations is given by the Euler formula. Imagine we started out with the seond equation $\ddot{x} = 25x$ and found the solutions e^{5t} and e^{-5t} as we just did. Now, a way to solve $\ddot{x} = -25x$ from there would be to observe that we could insert a complex i in the exponential. We observe

$$\begin{aligned} \tfrac{d^2}{dt^2}e^{5t} &= 25\,e^{5t} \ , & \tfrac{d^2}{dt^2}e^{-5t} &= (-5)^2\,e^{-5t} = 25\,e^{-5t} \ , \\ \tfrac{d^2}{dt^2}e^{5it} &= (5i)^2\,e^{5it} = -25\,e^{5it} \ , & \tfrac{d^2}{dt^2}e^{-5it} &= (-5i)^2\,e^{-5it} = -25\,e^{-5it} \ . \end{aligned}$$

Thus the solutions e^{5t} and e^{-5t} of the ODE $\ddot{x} - 25x = 0$ led us to the solutions e^{5it} and e^{-5it} of the ODE $\ddot{x} + 25x = 0$. The functions e^{5it} and e^{-5it} are complex – but we were looking for real solutions. Remember that by the Euler formula

$$e^{\pm 5it} = \cos(5t) \pm i\sin(5t) \ .$$

We see that the real part is $\cos(5t)$ (for both) and the imaginary part is $\pm\sin(5t)$. So the real and the imaginary part of *either* one of the solutions e^{5it} and e^{-5it} give us back the solution we originally found.