Recitation 8, March 7, 2006

Homogeneous second order linear equations

Solution suggestions

1. Find two independent real solutions of $\ddot{x} + 4\dot{x} + 3x = 0$.

Ans. The characteristic polynomial $p(s) = s^2 + 4s + 3 = (s+1)(s+3)$ has roots $-1, -3$, so the equation has two exponential solutions, e^{-t} and e^{-3t}. This is "overdamped."

2. Find two independent real solutions of $\ddot{x} + 4\dot{x} + 5x = 0$.

Ans. The characteristic polynomial $p(s) = s^2 + 4s + 5$ has roots $-2 \pm i$, so the equation has two exponential solutions, $e^{(-2+i)t}$ and $e^{(-2-i)t}$. Real solutions are obtained by taking the real and imaginary parts of either one: $e^{-2t}\cos t$ and $e^{-2t}\sin t$. This is "underdamped."

3. Find two independent real solutions of $\ddot{x} + 4\dot{x} + 4x = 0$. Check that they are solutions.

Ans. The characteristic polynomial $p(s) = s^2 + 4s + 4$ has just one root, -2. There is just one exponential solution, e^{-2t}. A second solution is obtained by multiplying by t: te^{-2t}. Let's check this:

$$\begin{array}{rrcl} 4] & x & = & te^{-2t} \\ 4] & \dot{x} & = & (1-2t)e^{-2t} \\ 1] & \ddot{x} & = & (-2-2(1-2t))e^{-2t} = (-4+4t)e^{-2t} \\ \hline & & & (4t + 4(1-2t) + (-4+4t))e^{-2t} = 0 \end{array}$$

This is "critically damped."

4. In each case, find the solution with $x(0) = 0$, $\dot{x}(0) = 2$.

Ans. (1) $x = c_1e^{-t} + c_2e^{-3t}$, $\dot{x} = -c_1e^{-t} - 3c_2e^{-3t}$, so $0 = x(0) = c_1 + c_2$, $2 = \dot{x}(0) = -c_1 - 3c_2$. Add them to get $2 = -2c_2$, $c_2 = -1$, $c_1 = 2$, $x = 2e^{-t} - e^{-3t}$.

(2) $x = e^{-2t}(a\cos t + b\sin t)$, so $0 = x(0) = a$ and $x = be^{-2t}\sin t$. Then $\dot{x} = be^{-2t}(\cos t - 2\sin t)$, so $2 = \dot{x}(0) = b$: $x = 2e^{-2t}\sin t$.

(3) $x = (at + b)e^{-2t}$, $0 = x(0) = b$, so $x = ate^{-2t}$. Then $\dot{x} = a(1-2t)e^{-2t}$ and $2 = \dot{x}(0) = a$: $x = 2te^{-2t}$.

5. Explain why e^{r_1t} and e^{r_2t} are "linearly independent" functions whenever $r_1 \neq r_2$. This means: neither one is a constant multiple of the other. [Argue by "contradiction": suppose one was a constant multiple of the other. What would follow?]

Ans. There are many ways to see this. Here's one. Suppose $e^{r_1t} = ce^{r_2t}$. Divide through by e^{r_2t} (which is legitimate because this function never vanishes): $e^{(r_1-r_2)t} = c$ for all t. If we differentiate this we find $(r_1 - r_2)e^{(r_1-r_2)t} = 0$. If then set $t = 0$ we find $r_1 - r_2 = 0$, which it was supposed not to be. Conclusion: There's no such c. By swapping r_1 and r_2 we see that you can't write e^{r_2t} as a constant multiple of r^{r_1t} either.

6. Explain why a nonzero solution to the equation in **(1)** can have at most one critical point (i.e. there's at most one value of t for which $\dot{x}(t) = 0$). Ditto with solutions to **(3)**.

Ans. The general solution of **(1)** is $x = c_1 e^{-t} + c_2 e^{-3t}$. We want to see that x has at most one critical point if it's not the zero solution, i.e. provided c_1 and c_2 are not both zero. If $c_1 = 0$, then $x = c_2 e^{-3t}$, which has no critical points: OK. If $c_2 = 0$, then $x = c_1 e^{-t}$, which has no critical points: OK. If neither c_1 nor c_2 is zero, let's look at the derivative $\dot{x} = -c_1 e^{-t} - 3c_2 e^{-3t}$. To find the critical points of x, set $\dot{x}(t) = 0$ and try to solve for t: $c_1 e^{-t} = -c_2 e^{-3t}$. Divide through by e^{-3t} and by c_1: $e^{2t} = -c_2/c_1$. This has no solutions in t if c_1 and c_2 are of the same sign, and exactly one if they are of opposite sign.

The general solution of **(2)** is $x = (at + b)e^{-2t}$. This is the zero function when $a = b = 0$, so suppose they aren't both zero. Compute $\dot{x} = (a - 2(at + b))e^{-2t} = (-2at + a - 2b)e^{-2t}$. If this is zero we can divide through by e^{-2t} to find $2at = a - 2b$. This has no solutions if $a = 0$ (since then $b \neq 0$), and one if $a \neq 0$ (namely $t = (a - 2b)/2a$).