Recitation 10, March 14, 2006

Operators, Exponential Response, Exponential shift, Undetermined Coefficients

Solution suggestions

1. What is the general solution of $\frac{d^4x}{dt^4} - x = 0$?

Ans. Plugging e^{rt} into $\frac{d^4x}{dt^4} - x = 0$ we obtain the characteristic equation

$$r^4 - 1 = 0 \ .$$

The roots are the 4th roots of 1 which are $e^{\frac{i\pi}{2}k}$ for $k = 0..3$. Thus we obtain $r = 1, i, -1, -i$. The basic real solutions are $e^{-t}, \cos t, \sin t, e^t$. Notice that the real and imaginary part of the root $r = i$ already gives to real solutions, and that the root $r = -i$ gives the same real solutions (up to a sign in the imaginary part). Thus, the general solution is

$$x(t) = c_1 e^{-t} + c_2 e^t + c_3 \cos t + c_4 \sin t = c_1 e^{-t} + c_2 e^t + A \cos(t - \phi) \ .$$

2. What is an exponential solution of $\frac{d^4x}{dt^4} - x = e^{-2t}$?

Ans. Using the ERF $\frac{A}{p(r)} e^{rt}$ is a solution to $p(D)x = Ae^{rt}$ unless $p(r) = 0$. In our case $p(r) = r^4 - 1$. We check $p(-2) = 15$. Thus, with $A = 1$ and $r = -2$ we have found the particular solution

$$x_p(t) = \frac{A}{p(r)} e^{rt} = \frac{1}{15} e^{-2t} \ .$$

3. What is a sinusoidal solution of $\frac{d^4x}{dt^4} - x = \cos(2t)$?

Ans. We look at the complex ODE $\frac{d^4z}{dt^4} - z = e^{2it}$ and realize that the desired solution is the real part of a solution to this complex ODE, i.e. $x(t) = \operatorname{Re} z(t)$. To the complex ODE we apply the ERF. We check that with $p(r) = r^4 - 1$ and $r = 2i$ we find $p(2i) = 15$. Thus, with $A = 1$ and $r = 2i$ we have found the complex particular solution

$$z_p(t) = \frac{A}{p(r)} e^{rt} = \frac{1}{15} e^{2it} \ .$$

The real part is $x_p(t) = \frac{1}{15} \cos 2t$.

4. What is a polynomial solution of $\frac{d^2x}{dt^2} - x = t^2 + t + 1$?

$-1]$	x	$=$	at^2	$+$	bt	$+$	c
$0]$	$\dot{x}$	$=$			$2at$	$+$	b
$1]$	$\ddot{x}$	$=$					$2a$
	t^2+t+1	$=$	$-at^2$	$-$	bt	$+$	$(-c+2a)$

so $a=-1$, $b=-1$, $c=-1$ and $x_p=-t^2-t-1$.

5. Compute $p(D)\cos(2t)$ if $p(s)=s^4-1$. How does this relate to **(3)**?

Ans. By the ESL we have

$$p(D)\, e^{st} = e^{st}\, p(D+s)\, 1 = e^{st} p(s)\ .$$

With $s=2i$ and $p(2i)=15$ we obtain

$$p(D)\, e^{2it} = 15\, e^{2it}\ .$$

Now we take the real part and obtain on the RHS just $15\cos(2t)$ and on the LHS

$$\mathrm{Re}\left(p(D)\, e^{2it}\right) = p(D)\ \mathrm{Re}\left(e^{2it}\right) = p(D)\cos(2t)\ .$$

Therefore, we have found

$$p(D)\cos(2t) = 15\cos(2t)\ .$$

The function $\frac{1}{15}\cos(2t)$ is exactly the particular solution constructed in **(3)**.

6. Compute $D^2(e^{2t}\cos(t))$ in three ways: (1) directly; (2) by writing the function as the real part of an exponential; and (3) using the exponential shift law.

Ans. First we compute it directly:

$$D^2\Big(f(t)g(t)\Big) = \ddot{f}(t)\, g(t) + 2\dot{f}(t)\ \dot{g}(t) + f(t)\ \ddot{g}(t)\ .$$

Thus, we have

$$D^2(e^{2t}\cos(t)) = 4e^{2t}\cos(t) - 4e^{2t}\sin(t) - e^{2t}\cos(t) = 3e^{2t}\cos(t) - 4e^{2t}\sin(t)\ .$$

We can also write the function $e^{2t}\cos(t)$ as the real part of $e^{\alpha t}$ with $\alpha = 2+i$. Thus,

$$\begin{aligned} D^2\ \mathrm{Re}\, e^{\alpha t} &= \mathrm{Re}\left(D^2\ e^{\alpha t}\right) = \mathrm{Re}\left(\alpha^2 e^{\alpha t}\right) \\ &= \mathrm{Re}\left((3+4i)e^{(2+i)t}\right) = 3e^{2t}\cos(t) - 4e^{2t}\sin(t)\ . \end{aligned}$$

Finally by the exponential shift law we write

$$\begin{aligned} D^2(e^{2t}\cos(t)) &= e^{2t}\ (D+2I)^2\cos t = e^{2t}\ (D^2+4D+4I)\cos t \\ &= e^{2t}\ (-\cos t - 4\sin t + 4\cos t) = 3e^{2t}\cos(t) - 4e^{2t}\sin(t)\ . \end{aligned}$$

7. What is a solution of $\ddot{x}+9x=\cos(3t)$? Do this by replacing the signal with a complex exponential signal and using the "resonant exponential response formula."

Ans. We look at the complex ODE $\ddot{z} + 9z = e^{3it}$ and realize that the desired solution is the real part of a solution to this complex ODE, i.e. $x(t) = \operatorname{Re} z(t)$. To the complex ODE we apply the "resonant exponential response formula". In fact, we check that $p(r) = r^2 + 9$ and $p(3i) = 0$. We compute $p'(r) = 2r$ and $p'(3i) = 6i \neq 0$. By the "resonant exponential response formula" with $A = 1$ a particular solution is provided by

$$z_p(t) = \frac{1}{p'(3i)} t e^{3it} = \frac{1}{6i} t e^{3it} \, .$$

Taking its real part we obtain $x_p(t) = \frac{1}{6} t \sin t$.