Recitation 11, March 16, 2006

Superposition, Frequency Response

Solution suggestions

1. Find a particular solution for $\ddot{x} + 4x = \sin(2t)$.

Ans. We write the complex ODE $\ddot{z} + 4z = e^{2it}$ and realize that the desired particular solution is the imaginary part of a particular solution to this complex ODE. Plugging e^{rt} into the differential equation we obtain the chatacteristic equation $p(r) = r^2 + 4$, thus $p(2i) = 0$. We compute $p'(r) = 2r$ and $p'(2i) = 4i$. Therefore, by the resonant exponential response formula we have the particular solution

$$z_p(t) = \frac{te^{2it}}{p'(2i)} = -\frac{ite^{2it}}{4} \ .$$

We obtain for the imaginary part $x_p(t) = -\frac{1}{4}t\cos(2t)$.

2. Find a particular solution for $\ddot{x} + 4x = 1$, and then find the general solution for $\ddot{x} + 4x = 3 + 2\sin(2t)$.

Ans. By pulgging $x_p = A$ into the ODE with $\dot{x}_p = 0$ we obtain $x_p = A = \frac{1}{4}$. Now, to find the general solution of $\ddot{x} + 4x = 3 + 2\sin(2t)$ we use the superposition principle: we have to add thrice the particular solution for $\ddot{x} + 4x = 1$ and twice the particular solution for $\ddot{x} + 4x = \sin(2t)$ to the general solution of $\ddot{x} + 4x = 0$. To construct the latter, we recall that the characteristic polynomial $p(r) = r^2 + 4$ has roots $\pm 2i$. The real and imaginary part of e^{2it} gives to independent solutions $\cos(2t)$ and $\sin(2t)$. Thus, the general solution to $\ddot{x} + 4x = 0$ is $c_1\cos(2t) + c_2\sin(2t)$. Putting all the pieces together we obtain

$$x(t) = \frac{3}{4} - \frac{1}{2}t\cos(2t) + c_1\cos(2t) + c_2\sin(2t) \ .$$

The following problems consider the frequency response of the first order equation $\dot{x} + x = \cos(\omega t)$.

3. Express the amplitude of the sinusoidal system response to $\dot{x} + x = \cos(\omega t)$ as a function of ω.

Ans. We write the complex ODE $\ddot{z} + 4z = e^{i\omega t}$ and realize that the desired particular solution is the real part of a particular solution to this complex ODE. Plugging e^{rt} into the ODE we obtain the characteristic equation $p(r) = r + 1$, thus $p(i\omega) = 1 + i\omega$. This can never be zero as the real part is always 1. By the exponential response formula a particular solution to the complex ODE is given by

$$z_p(t) = \frac{1}{p(i\omega)}e^{i\omega t} = \frac{1}{1 + i\omega}e^{i\omega t} \ .$$

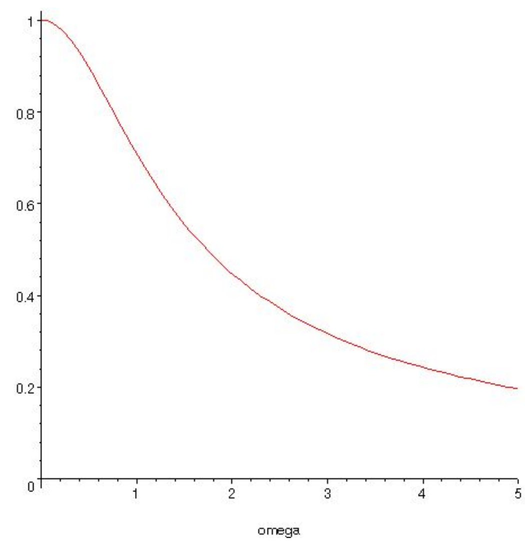
If we write the complex number $1+i\omega$ is polar coordinates we obtain $1+i\omega = \sqrt{1+\omega^2}e^{i\phi}$ with $\tan\phi = \omega$. The particular solution then becomes

$$z_p(t) = \frac{1}{\sqrt{1+\omega^2}e^{i\phi}}e^{i\omega t} = \frac{1}{\sqrt{1+\omega^2}}e^{i(\omega t-\phi)} \ .$$

Taking the real part, we obtain $x_p(t) = \frac{1}{\sqrt{1+\omega^2}}\cos(\omega t - \phi)$. The amplitude $A(\omega)$ and phase lag ϕ are

$$A(\omega) = \frac{1}{\sqrt{1+\omega^2}} \ , \qquad \tan\phi = \omega \ .$$

4. Sketch the graph of this function of ω (for $\omega \geq 0$).



5. At what circular frequency is the amplitude of the system response just half that of the input signal?

Ans. We are looking for ω such that $A(\omega) = \frac{1}{2}$. Thus, $1+\omega^2 = 4$ or $\omega = \sqrt{3}$.

6. What is the value of ω for which the phase lag is $\pi/4$?

Ans. We are looking for ω such that $\omega = \tan\phi = 1$.