Recitation 13, March 23, 2006

Fourier Series: Introduction

Solutions suggestions

1. What is the general solution to $\ddot{x} + \omega_n^2 x = 0$? [Quick!]

Ans. The characteristic polynomial is $p(s) = s^2 + \omega_n^2$. The roots are $\pm i\omega_n$ and the complex solutions $e^{\pm i\omega_n t}$. We pick one of the complex solutions and determine its real and imaginary part. The general solution is

$$x = c_1 \cos(\omega_n t) + c_2 \sin(\omega_n t) \ .$$

2. Discuss why

$$\ddot{x} + \omega_n^2 x = a\cos(\omega t) \quad \text{has solution} \quad x_p = a\,\frac{\cos(\omega t)}{\omega_n^2 - \omega^2}$$

$$\ddot{x} + \omega_n^2 x = b\sin(\omega t) \quad \text{has solution} \quad x_p = b\,\frac{\sin(\omega t)}{\omega_n^2 - \omega^2}$$

Ans. We have to find a particular solution of the complex ODE $\ddot{z} + \omega_n^2 z = Ae^{i\omega t}$. Then its real part $\operatorname{Re} z_p(t)$ is a solution for the first ODE (with A=a), and its imaginary part $\operatorname{Im} z_p(t)$ is a solution for the second ODE (with A=b).The characteristic polynomial is $p(s) = \omega_n^2 + s^2$. We find $p(i\omega) = \omega_n^2 - \omega^2$. Since circular frequency are always positive $p(i\omega)$ is zero exactly if $\omega = \omega_n$.

Case 1: ($\omega \neq \omega_n$) Now $p(i\omega) = \omega_n^2 - \omega^2 \neq 0$ and we can write down a particular solution by the exponential response formula:

$$z_p(t) = \frac{A}{\omega_n^2 - \omega^2} e^{i\omega t} \ .$$

The real and the imaginary part are exactly the real particular solutions for the two ODEs.

$$\ddot{x} + \omega_n^2 x = A\cos(\omega t) \quad \text{has solution} \quad x_p = A\,\frac{\cos(\omega t)}{\omega_n^2 - \omega^2}$$

$$\ddot{x} + \omega_n^2 x = A\sin(\omega t) \quad \text{has solution} \quad x_p = A\,\frac{\sin(\omega t)}{\omega_n^2 - \omega^2}$$

Case 2: ($\omega = \omega_n$) Now $p(i\omega) = \omega_n^2 - \omega^2 = 0$ and $p'(i\omega) = 2i\omega$. By the resonant ERF the particular solution is given by

$$z_p(t) = \frac{At}{2i\omega} e^{i\omega t} \ .$$

The real and the imaginary part are exactly the real particular solutions for the two ODEs.

$$\ddot{x} + \omega x = A\cos(\omega t) \quad \text{has solution} \quad x_p = At\,\frac{\sin(\omega t)}{2\omega}$$

$$\ddot{x} + \omega x = A\sin(\omega t) \quad \text{has solution} \quad x_p = -At\,\frac{\cos(\omega t)}{2\omega}$$

3. For what values of ω_n is there a sinusoidal solution to $\ddot{x} + \omega_n^2 x = \sin(t)$? What is it when it exists? What is the general solution when no sinusoidal solution exists? Sketch the graph of one solution in that case.

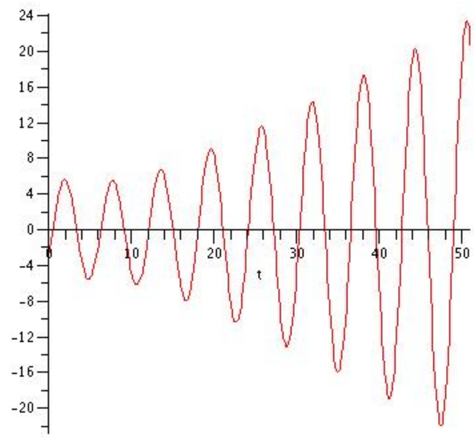
Ans. By **(3)** we know that the general solution for $\omega_n \neq 1$ is

$$x(t) = c_1\cos(\omega_n t) + c_2\sin(\omega_n t) + \frac{\sin t}{\omega_n^2 - 1}\ .$$

If $\omega_n = 1$ then the general solution becomes

$$x(t) = c_1\cos t + c_2\sin t + \frac{t\cos t}{2}\ .$$

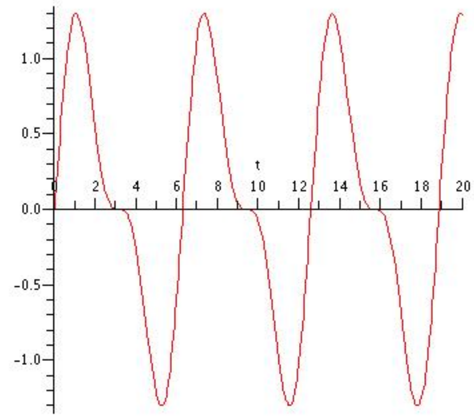
Here is the graph of $x(t) = 6\cos(t-2) + \frac{t\cos t}{2}$:



4. What is the period of $f(t) = \sin(t) + (1/2)\sin(2t)$?

[A function is *periodic* if there is a number $P > 0$ such that $f(t+P) = f(t)$ for all t. Such a number P is then a "period" of $f(t)$. If $f(t)$ is a periodic function which is continuous and not constant, then there is a smallest period, often called *the* period.]

Ans. We know that $\sin t$ has period 2π, i.e. $t \mapsto t + 2\pi$ leaves $\sin t$ unchanged for arbitrary t and 2π is the smallest such number. Accordingly, $\sin(2t)$ must have period π. Again, $t \mapsto t+\pi$ or $2t \mapsto 2t + 2\pi$ leaves $\sin(2t)$ unchanged for arbitrary t. Now, what is the period of their linear combination. It is the least common multiple of the two whi is 2π. Here is the graph



5. For what values of ω is there a periodic solution to $\ddot{x} + \omega_n^2 x = \sin(t) + (1/2)\sin(2t)$? What is it when such exists? What is the general solution for the values of ω when no periodic solution exists?

Ans. The characteristic polynomial is again $p(s) = \omega_n^2 + s^2$. Let us first assume that $\omega_n \neq 1$ and $\omega_n \neq 2$. By applying the ERF twice we can write down the general solution

$$x(t) = c_1 \cos(\omega_n t) + c_2 \sin(\omega_n t) + \frac{\sin t}{\omega_n^2 - 1} + \frac{\sin(2t)}{2\,(\omega_n^2 - 4)} \,.$$

The first two pieces have period $\frac{2\pi}{\omega_n}$, the last two pieces periods 2π and π. So, the period is the least common multiple of $\frac{2\pi}{\omega_n}$ and 2π.

Now, let us assume $\omega_n = 1$. The general solution is then given by

$$\begin{aligned} x(t) &= c_1 \cos(t) + c_2 \sin(t) - \frac{t \cos t}{2} + \frac{\sin(2t)}{2\,(1-4)} \\ &= c_1 \cos(t) + c_2 \sin(t) - \frac{t \cos t}{2} - \frac{\sin(2t)}{6} \,. \end{aligned}$$

Because of the third term this solution is not periodic.

Similarly, if $\omega_n = 2$. The general solution is then given by

$$x(t) = c_1 \cos(2t) + c_2 \sin(2t) + \frac{\sin t}{3} - \frac{t \cos(2t)}{8} \,.$$

Because of the fourth term this solution is not periodic.