# Recitation 15, April 6, 2006

## Fourier Series: Harmonic response

### Solution suggestions

**1.** Find the Fourier series for the function of period $2\pi$ which is given by $f(t) = t/\pi$ for $-\pi < t < \pi$.

**Ans.** The period of $f(t)$ is $2\pi$, thus $L = \pi$. As you can check we also have $f(-t) = -f(t)$. Therefore, $f(t)$ is odd. All the coefficients $a_n$ in the Fourier series must be zero, and we are left with computing the $b_n$'s, i.e.

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\,\sin(nx)\,dx \\ &= \frac{2}{\pi^2}\int_0^{\pi} x\,\sin(nx)\,dx\,. \end{aligned}$$

The integral can be carried out by integration by parts:

$$\begin{aligned} b_n &= \frac{2}{\pi^2}\int_0^{\pi} x\,\sin(nx)\,dx \\ &= \frac{2}{\pi^2}\Big[-x\frac{\cos(nx)}{n} + \frac{1}{n}\int \cos(nx)\,dx\Big]_0^{\pi} \\ &= \frac{2}{\pi^2}\Big[-x\frac{\cos(nx)}{n} + \frac{1}{n^2}\sin(nx)\Big]_0^{\pi} \\ &= \frac{2}{\pi^2}\left(-\frac{\pi\cos(n\pi)}{n}\right) = \frac{2(-1)^{n+1}}{\pi n}\,. \end{aligned}$$

Thus the Fourier series is

$$\begin{aligned} f(t) &= \frac{2}{\pi}\Big(\sin(t) - \frac{1}{2}\sin(2t) + \frac{1}{3}\sin(3t) + \dots\Big) \\ &= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nt)\,. \end{aligned}$$

**2.** Find the Fourier series for the function of circular frequency $\omega$ which is given by $g(t) = t/L$ for $-L < t < L$, where $L = \pi/\omega$.

**Ans.** The period of $g(t)$ is $2L = 2\pi/\omega$. The relation to the function $f(t)$ is given by

$$g(t) = f\left(\frac{\pi t}{L}\right)\,.$$

Using the Fourier series from **(1)** we obtain

$$g(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi t}{L}\right)\,.$$

**3.** Now drive the harmonic oscillator with the function $g(t)$ from **(2)**: $\ddot{x} + \omega_n^2 x = g(t)$. Express a periodic solution as a Fourier series.

**Ans.** Let's first look at the differential equation

$$\ddot{x} + \omega_n^2 x = b_k \sin(k\omega t) .$$

To obtain a particular solution we look at the complex equation

$$\ddot{z} + \omega_n^2 z = b_k e^{ik\omega t}$$

and remember that a solution of our original problem is given by the imaginary part of a particular complex solution, i.e. $x_p(t) = \operatorname{Im} z_p(t)$.

The characteristic polynomial is $p(s) = s^2 + \omega_n^2$. We already know what the particular complex solution then is: if $\omega_n \neq k\omega$ so that $p(ik\omega) \neq 0$, the particular solution is

$$z_p(t) = \frac{b_k e^{ik\omega t}}{p(ik\omega)} = \frac{b_k}{\omega_n^2 - k^2\omega^2} e^{ik\omega t} .$$

If $\omega = \omega_n$ then $p(ik\omega) = 0$, and the particular solution is instead

$$z_p(t) = \frac{b_k t e^{ik\omega t}}{p'(ik\omega)} = \frac{b_k t}{2ik\omega} e^{ik\omega t} .$$

Remember, that we still have to take the imaginary part, so the real particular solution of

$$\ddot{x} + \omega_n^2 x = b_k \sin(k\omega t) .$$

is given by

$$\begin{array}{lllcl} \text{if } \omega_n & \neq & k\omega : & x_p(t) & = & \frac{b_k}{\omega_n^2 - k^2\omega^2} \sin(k\omega t) , \\ \text{if } \omega_n & = & k\omega : & x_p(t) & = & -\frac{b_k}{2k\omega}\, t \cos(k\omega t) . \end{array}$$

By the superposition principle we can now also write down the particular solution to

$$\ddot{x} + \omega_n^2 x = \sum_{k=1}^{\infty} b_k \sin(k\omega t) .$$

All we have done here is replacing the input signal by a sum of input signals. All we have to do to find the new particular solution is to sum up the particular solutions for each input signal. This means that the particular solution becomes

$$x_p(t) = \sum_{k=1}^{\infty} \frac{b_k}{\omega_n^2 - k^2\omega^2} \sin(k\omega t)$$

as long as *all* $\omega_n \neq k\omega$. For the ODE with input signal $g(t)$

$$\ddot{x} + \omega_n^2 x = g(t) = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin\left(\frac{k\pi t}{L}\right)$$

we have

$$b_k = \frac{2}{k\pi}(-1)^{k+1} \qquad k\omega = \frac{k\pi}{L} \; .$$

The particular solution is

$$x_p(t) \;\; = \;\; \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k\left[\omega_n^2 - \left(\frac{k\pi}{L}\right)^2\right]} \sin\left(\frac{k\pi t}{L}\right)$$

as long as $\omega_n \neq \frac{k\pi}{L}$ for all $k$.

**4.** Imagine changing the capacitance in the AM turner; this changes $\omega_n$. For what values of $\omega_n$ does resonance occur?—that is, for what values of $\omega_n$ does there fail to be a periodic solution? When $\omega_n$ is near one of those values, what is the periodic solution like?

**Ans.** We see that if we change $\omega_n$ such that for a particular $k_0$ we will have $\omega_n = k_0\omega = \frac{k_0\pi}{L}$ then the particular solution we wrote down before is not valid any more. The piece in our particular solution that corresponded to the input $b_{k_0}\sin(k_0\omega t)$ now has to be the resonant solution

$$-\frac{b_{k_0}}{2k_0\omega}\, t\, \cos(k_0\omega t) \; .$$

Thus, if for $k_0$ we have $k_0\omega = \omega_n$ then the particular solution is

$$x_p(t) \;\; = \sum_{k=1\, but\, not\, k_0}^{\infty} \frac{b_k}{\omega_n^2 - k^2\omega^2}\sin(k\omega t) - \frac{b_{k_0}}{2k_0\omega}\, t\, \cos(k_0\omega t) \; .$$

We observe that because of the second term on RHS, $x_p(t)$ is no longer periodic. If $\omega_n \approx k_0\omega$ but not exactly equal, we still have the periodic particular solution

$$x_p(t) \;\; = \;\; \sum_{k=1}^{\infty} \frac{b_k}{\omega_n^2 - k^2\omega^2}\sin(k\omega t) \; .$$

But as $\omega_n^2 - k_0^2\omega^2$ will be very small compared to all the other denominators, the dominant term in this sum will be

$$x_p(t) \;\; \approx \;\; \frac{b_{k_0}}{\omega_n^2 - k_0^2\omega^2}\sin(k_0\omega t) \; .$$

Now since $k_0\omega \approx \omega_n$ we can write

$$\omega_n^2 - k_0^2\omega^2 = (\omega_n + k_0\omega)(\omega_n - k_0\omega) \approx 2k_0\omega(\omega_n - k\omega); ,$$

and the dominant part of the particular solution is

$$x_p(t) \;\; \approx \;\; \frac{b_{k_0}}{2k_0\omega(\omega_n - k_0\omega)}\sin(k_0\omega t) \; .$$

In the case of the input signal $g(t)$ we thus have the following result: if we change $\omega_n$ such that $\frac{\omega_n L}{\pi}$ becomes an integer – which we call $k_0$ – then the dominant part in the periodic solution is

$$\frac{2(-1)^{k_0+1}}{\pi k_0\left[\omega_n^2 - \left(\frac{k_0\pi}{L}\right)^2\right]}\sin\left(\frac{k_0\pi t}{L}\right) \approx \frac{L\,(-1)^{k_0+1}}{\pi^2 k_0^2\left[\omega_n - \frac{k_0\pi}{L}\right]}\sin\left(\frac{k_0\pi t}{L}\right) \; .$$