# 18.03 Recitation 17, April 13, 2006

## Convolution

### Solution suggestions

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$$

**1. (a)** What is the differential operator $p(D)$ whose weight function (i.e. unit impulse response) is the unit step function $u(t)$?

Verify that $u(t)*q(t)$ is the solution to $p(D)x = q(t)$ with rest initial conditions. (Since we are always interested only in $t > 0$ , we could write $1 * q(t)$ instead of $u(t) * q(t)$.)

**Ans.** We already know the relation between $u(t)$ and $\delta(t)$ from their definition. It is $Du(t) = \delta(t)$. Therefore, we should have $p(D) = D = d/dt$. Now, let us check that $u(t)$ really is the unit impulse response to

$$D\,w(t) = \delta(t)$$

with rest initial condition, i.e. $w(t) = 0$ for $t < 0$. From the lecture we know that for $t > 0$ this is the same as

$$D\;w(t) = 0$$

with initial condition $w(0) = 1$. The solutions $w(t)$ are simply constants $w(t) = a$. The initial condition $w(0) = 1$ fixes this constant to be $a = 1$. This means for $t > 0$ the solution is $w(t) = 1$.

Now we want to solve

$$p(D)\;x(t) = \dot{x}(t) = q(t)\;.$$

with initial condition $x(0) = 0$. We integrate both sides and obtain the solution

$$x(t) = x(0) + \int_0^t q(\tau)\,d\tau = \int_0^t q(\tau)\,d\tau\;.$$

Now for $t > 0$ we have

$$u(t) * q(t) = \int_0^t u(t-\tau)\,q(\tau)\,d\tau = \int_0^t q(\tau)\,d\tau\;.$$

For the last step, we have used that $u(t-\tau) = 1$ as long as $\tau$ is in between 0 and $t$. Thus, we have found

$$x(t) = u(t) * q(t)\;.$$

**(b)** What is the differential operator $p(D)$ whose weight function is $u(t)t$?

Verify that $t * t^n$ is the solution, with rest initial conditions, to $p(D)x = t^n$.

**Ans.** Again, we have to find the unit impulse response with rest initial conditions, i.e.

$$p(D)w(t) = \delta(t) \ .$$

For $t > 0$ this means $p(D)w(t) = 0$, and we want $w(t)$ to turn out to be $tu(t) = t$. So what differential operator has as a general homogeneous solution $t$ or better $at + b$? It's $p(D) = D^2$. We see that the general solution to $p(D)w(t) = 0$ is in fact $w(t) = at + b$. The initial conditions $w(0) = 0$ and $\dot{w}(0) = 1$ then fix $a = 1$ and $b = 0$.

Now, for $t > 0$ let's look at

$$D^2 x = t^n$$

with initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$. Integrating once gives

$$\dot{x}(t) = \dot{x}(0) + \int_0^t \tau^n \, d\tau = \dot{x}(0) + \frac{1}{n+1} t^{n+1} \ ,$$

integrating again gives

$$x(t) = x(0) + t\dot{x}(0) + \frac{1}{(n+1)(n+2)} t^{n+2} \ .$$

With the initial condition we obtain $x(t) = \frac{1}{(n+1)(n+2)} t^{n+2}$.

On the other hand we have from the definition

$$t * t^n = \int_0^t (t-\tau)\tau^n \, d\tau = t \int_0^t \tau^n \, d\tau - \int_0^t \tau^{n+1} \, d\tau = \frac{t^{n+2}}{n+1} - \frac{t^{n+2}}{n+2} = \frac{t^{n+2}}{(n+1)(n+2)} \ .$$

which is exactly $x(t)$.

**2. (a)** Suppose $a \geq 0$. Figure out what $w(t) * \delta(t-a)$ is by using the fact that it is the solution to the equation $p(D)x = \delta(t-a)$ with rest initial conditions.

**Ans.** Let us look at the differential equation

$$p(D)\, x = \delta(t-a)$$

with rest initial conditions which are $x(t) = 0$ for $t < a$. As long as $t < a$ we have no input signal, thus the output just remains zero, so really $w(t) = 0$ for $t < a$. Let's define a new time by $\tau = t - a$. We check $d/dt = d/d\tau$. Thus, it's the same to look for a solution of

$$p(D)\, x = \delta(\tau)$$

with rest initial conditions. But this is just the definition of weight function, i.e. the solution is $w(\tau)$ with $w(\tau) = 0$ for $\tau < 0$. Since $\tau = t - a$ the function $w(t-a)$ is the solution to the original problem.

**(b)** Then figure out what $w(t) * \delta(t-a)$ is using the definition.

**Ans.** We have to compute

$$w(t) * \delta(t-a) = \int_0^t w(t-\tau)\, \delta(\tau - a)\, d\tau$$

Now, $\delta(\tau - a)$ is zero unless $\tau = a$. In general we have

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)\;.$$

Therefore, we obtain

$$w(t) * \delta(t-a) = w(t-a)\;,$$

which agrees with **(a)**.

**3.** Compute the convolution product $e^{-t} * (1 + \cos(t))$ by using the integral.
**Ans.** We have to compute

$$e^{-t} * (1+\cos(t)) = \int_0^t e^{-(t-\tau)}(1+\cos(\tau))\,d\tau\;.$$

By using Euler's formula we can write the integral as

$$\int_0^t e^{-(t-\tau)}(1+\cos(\tau))\,d\tau = e^{-t}\Big[\int_0^t e^{\tau} + \int_0^t \mathrm{Re}\,(e^{(1+i)\tau})\,d\tau\Big]\;.$$

This can be evaluated as follows

$$\begin{aligned}
& e^{-t}\Big[\int_0^t e^{\tau} + \int_0^t \mathrm{Re}\,(e^{(1+i)\tau})\,d\tau\Big] \\
= \;& e^{-t}\Big[e^{\tau}\Big|_0^t + \mathrm{Re}\Big(\int_0^t e^{(1+i)\tau}\,d\tau\Big)\Big] \\
= \;& 1 - e^{-t} + e^{-t}\mathrm{Re}\Big(\frac{1}{1+i}e^{(1+i)\tau}\Big|_0^t\Big) \\
= \;& 1 - e^{-t} + e^{-t}\mathrm{Re}\Big(\frac{1}{1+i}e^{(1+i)t} - \frac{1}{1+i}\Big) \\
= \;& 1 - e^{-t} + e^{-t}\mathrm{Re}\Big(\frac{1}{\sqrt{2}e^{i\pi/4}}e^{(1+i)t} - \frac{1-i}{2}\Big) \\
= \;& 1 - \frac{3}{2}e^{-t} + \frac{1}{\sqrt{2}}\cos\Big(t - \frac{\pi}{4}\Big)\;.
\end{aligned}$$