18.03 Recitation 20, April 27, 2006

Systems of first order equations

Vocabulary: System of ODEs; Linear: time-independent, homogeneous; Matrix, matrix multiplication; Solutions, initial conditions; Autonomous equations; Vector field, phase plane, trajectory; Equilibrium solutions; Companion matrix.

Notation. $\mathbf{i}$ is the unit vector pointing east, and $\mathbf{j}$ is the unit vector pointing north. If $\mathbf{u}(t)$ is a parametrized curve in the plane, with coordinates $x(t)$ and $y(t)$, then $\dot{\mathbf{u}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j}$ is the velocity vector, tangent to the curve. The speed of the curve at time t is given by $||\dot{\mathbf{u}}(t)|| = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}$.

1. Define a vector field in the plane by putting the vector $y\mathbf{i} - x\mathbf{j}$ at the position (x, y). Sketch enough values of this vector field to visualize it and describe it in words. Then sketch a curve which is everywhere tangent to it and passes though the point $(1, 0)$.

The curve you drew is the trajectory of a solution of the system of ODEs

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$$

Solve this system of equations in the following way: substitute $\dot{y} = -x$ into the equation you get for $\ddot{x}$ by differentiating $\dot{x} = y$. This gives you a second order LTI ODE. The initial conditions for x and y give initial conditions for this new ODE. Solve it.

Then graph x against t, graph y against t, and plot the path of the curve $\mathbf{u}(t)$ in the plane. Does it look right?

2. Now reverse engineer this, starting with the second order IVP

$$\ddot{x} + (1/2)\dot{x} + (17/16)x = 0,$$

with initial condition $x(0) = 1$, $\dot{x}(0) = 0$. Use $y = \dot{x}$ for one of the pair of equations. So: write an equation for $\dot{y}$ in terms of x and y. Together this pair of equations determines $\dot{\mathbf{u}}$ in terms of $\mathbf{u}$. Solve the original second order ODE, and reinterpret your solution as a solution of the system you produced. Sketch graphs of x and of y as functions of t, and sketch the path of the curve $\mathbf{u}(t)$ (its "trajectory"). If you think of the variable x in the original equation as position, how is velocity, $\dot{x}$, represented in the picture of the trajectory?

3. Practice in matrix multiplication: Compute the following products:

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & u \\ y & v \end{bmatrix}.$$