18.03 Recitation 21, May 2, 2006

Eigenvalues and Eigenvectors

Solution suggestions

We'll solve the system of equations $\left\{ \begin{array}{rcl} \dot{x} & = & -5x - 3y \\ \dot{y} & = & 6x + 4y \end{array} \right.$

1. Write down the matrix of coefficients, A, so that we are solving $\dot{\mathbf{u}} = A\mathbf{u}$. What is its trace? Its determinant? Its characteristic polynomial $p_A(\lambda) = \det(A - \lambda I)$? Relate the trace and determinant to the coefficients of $p_A(\lambda)$.

Ans. If we think of the vector $\mathbf{u}(t)$ having the components $x(t)$ and $y(t)$ we can write

$$\mathbf{u}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} .$$

The velocity vector $\dot{\mathbf{u}}$ then is

$$\dot{\mathbf{u}}(t) = \dot{x}(t)\mathbf{i} + \dot{y}(t)\mathbf{j} = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} .$$

This means we can write the equations for x, y as

$$\begin{aligned} \dot{\mathbf{u}}(t) &= \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} -5 & -3 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} -5 & -3 \\ 6 & 4 \end{bmatrix}}_{=A} \mathbf{u}(t) . \end{aligned}$$

We compute

$$\operatorname{tr} \begin{bmatrix} -5 & -3 \\ 6 & 4 \end{bmatrix} = -5 + 4 = -1 , \qquad \det \begin{bmatrix} -5 & -3 \\ 6 & 4 \end{bmatrix} = -5 \cdot 4 - (-3) \cdot 6 = -2 .$$

Now we compute the characteristic polynomial

$$\begin{aligned} p_A(\lambda) &= \det(A - \lambda I) = \det \left(\begin{bmatrix} -5 & -3 \\ 6 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) \\ &= \det \begin{bmatrix} -5 - \lambda & -3 \\ 6 & 4 - \lambda \end{bmatrix} = (-5 - \lambda)(4 - \lambda) - (-3)6 \\ &= \lambda^2 + \lambda - 2 \\ &= \boldsymbol{\lambda^2} - \operatorname{tr}(\mathbf{A})\, \lambda + \det(\mathbf{A}) . \end{aligned}$$

2. Find the eigenvalues and then for each eigenvalue find a nonzero eigenvector.

Ans. The eigenvalues are the roots of $p_A(\lambda)$. The solutions of $\lambda^2 + \lambda - 2 = 0$ are

$$\lambda_1 = -\frac{1}{2} + \sqrt{\frac{1}{4} + 2} = -\frac{1}{2} + \frac{3}{2} = 1 , \quad \lambda_2 = -\frac{1}{2} - \sqrt{\frac{1}{4} + 2} = -2 ,$$

and thus $p_A(\lambda) = (\lambda-1)(\lambda+2)$. For $\lambda_1 = 1$, a nonzero eigenvector $\mathbf{v_1}$ with components a_1, b_1 must satisfy the equation

$$A \cdot \mathbf{v_1} = \lambda_1 \, \mathbf{v_1} = \mathbf{v_1} \ ,$$

or written in components it is

$$\begin{bmatrix} -5 & -3 \\ 6 & 4 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} .$$

We can write these two equations as

$$\begin{aligned} -6a_1 - 3b_1 &= 0 \ , \\ 6a_1 + 3b_1 &= 0 \ . \end{aligned}$$

The second equation is just the negative of the first one. Therefore, we get $2a_1 = -b_1$ as our only equation. If we pick $b_1 = 2$, then $a_1 = -1$, and the eigenvector is

$$\mathbf{v_1} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} .$$

However, any choice

$$\begin{bmatrix} -c \\ 2c \end{bmatrix}$$

with c a real number, but not zero would have been fine, too.

For $\lambda_2 = -2$ a nonzero eigenvector $\mathbf{v_2}$ with components a_2, b_2 must satisfy

$$A \cdot \mathbf{v_2} = \lambda_2 \, \mathbf{v_2} = -2\mathbf{v_2} \ ,$$

or written in components it is

$$\begin{bmatrix} -5 & -3 \\ 6 & 4 \end{bmatrix} \cdot \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} -2a_2 \\ -2b_2 \end{bmatrix} .$$

We can write these two equations as

$$\begin{aligned} -3a_2 - 3b_2 &= 0 \ , \\ 6a_2 + 6b_2 &= 0 \ . \end{aligned}$$

The second equation is just twice the negative of the first one. Therefore, we get $a_2 = -b_2$ as our only equation. If we pick $b_2 = 1$ then $a_1 = -1$, and the eigenvector is

$$\mathbf{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} .$$

However, any choice

$$\begin{bmatrix} -c' \\ c' \end{bmatrix}$$

with c' a real number but not zero would have been fine, too.

3. Draw the eigenlines and discuss the solutions whose trajectories live on each. Explain why each eigenline is made up of three distinct non-intersecting trajectories. Begin to construct a phase portrait by indicating the direction of time on portions of the eigenlines. Pick a nonzero point on an eigenline and write down all the solutions to $\dot{\mathbf{u}} = A\mathbf{u}$ whose trajectories pass through that point.

Ans. The eigenlines are the lines which run through the orgin and have a constant velocity vector $\mathbf{v_1}$ or $\mathbf{v_2}$. For $\lambda_1 = 1$ (blue graph in the picture below) we take

$$\mathbf{u}(t) = c\, e^{\lambda_1 t}\mathbf{v_1} = c\, e^t \begin{bmatrix} -1 \\ 2 \end{bmatrix} = e^t \begin{bmatrix} -c \\ 2c \end{bmatrix}$$

we can check that it is a solution to the ODE since

$$\dot{\mathbf{u}}(t) = \lambda_1 \mathbf{u}(t) = A \cdot \mathbf{u}(t) \ ,$$

and the initial condition is

$$\mathbf{u}(0) = \begin{bmatrix} -c \\ 2c \end{bmatrix} .$$

We distinguish the following three cases: (1) For $c > 0$, we see that $\mathbf{u}(0)$ is located in the second quadrant. If we let $t \to \infty$ we find a trajectory that runs away from the origin on the eigenline which lies in the second quadrant. If we let $t \to -\infty$ we find a trajectory that asymptotically approaches the origin on the eigenline which lies in the second quadrant. Just check that the factor e^t in fornt of $\mathbf{v_1}$ which multiplies the two components becomes very small as $t \to -\infty$. However, as e^t is never zero for finite t, the trajectory doesn't reach the origin in finite time.

(2) For $c < 0$, we see that $\mathbf{u}(0)$ is located in the fourth quadrant. If we let $t \to \infty$ we find a trajectory that runs away from the origin on the eigenline which lies in the fourth quadrant. If we let $t \to -\infty$ we find a trajectory that asymptotically approaches the origin on the eigenline which lies in the fourth quadrant.

(3) For $c = 0$ the vector is just the zero vector. The 'trajectory' is just the point staying at the origin for all times.

These three trajectories do not intersect.

For $\lambda_2 = -2$ (red graph in the picture below) we take

$$\mathbf{u}(t) = c' e^{\lambda_2 t}\mathbf{v_2} = c' e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = e^{-2t} \begin{bmatrix} -c' \\ c' \end{bmatrix}$$

we can check that

$$\dot{\mathbf{u}}(t) = \lambda_2 \mathbf{u}(t) = A \cdot \mathbf{u}(t) \ ,$$

and

$$\mathbf{u}(0) = \begin{bmatrix} -c' \\ c' \end{bmatrix} .$$

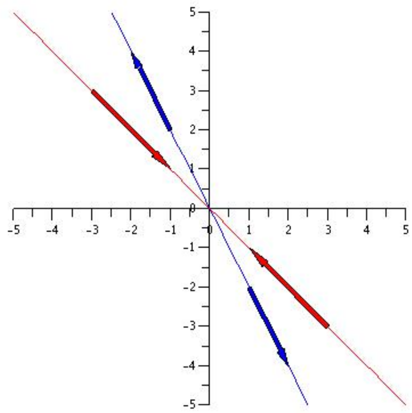
We distinguish three cases: (1) For $c' > 0$, we see that $\mathbf{u}(0)$ is located in the second quadrant. If we let $t \to \infty$ we find a trajectory that runs towards the origin on the eigenline which lies in the second quadrant. If we let $t \to -\infty$ we find a trajectory that runs away from the origin on the eigenline which lies in the second quadrant.

(2) For $c' > 0$, we see that $\mathbf{u}(0)$ is located in the fourth quadrant. If we let $t \to \infty$ we find a trajectory that runs towards the origin on the eigenline which lies in the fourth quadrant. If we let $t \to -\infty$ we find a trajectory that runs away the origin on the eigenline which lies in the fourth quadrant.

(3) For $c' = 0$ the vector is just the zero vector. The 'trajectory' is just the point staying at the origin for all times.

These three trajectories do not intersect.

We have the following picture:



4. Now study the solution $\mathbf{u}(t)$ such that $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Write $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as a linear combination of a vector from the first eigenline and a vector from the second eigenline. Use this decomposition to express the solution, and sketch its trajectory. Fill out the phase portrait.

Ans. We want to write

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \alpha \mathbf{v_1} + \beta \mathbf{v_2} = \alpha \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 1 \end{bmatrix} .$$

From the second component we see $2\alpha + \beta = 0$ or $\beta = -2\alpha$. Looking at the first component then gives $1 = -\alpha - \beta = \alpha$. Going back, we get $\beta = -2$. Thus,

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{v_1} - 2\mathbf{v_2} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} .$$

Now we write

$$\mathbf{u}(t) = e^{\lambda_1 t} \mathbf{v_1} - 2e^{\lambda_2 t} \mathbf{v_2} .$$

We can check that we have

$$\mathbf{u}(0) = \mathbf{v_1} - 2\mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} ,$$

as intended. Moreover,

$$\dot{\mathbf{u}}(t) = \lambda_1 \Big(e^{\lambda_1 t} \mathbf{v_1} \Big) + \lambda_2 \Big(-2e^{\lambda_2 t} \mathbf{v_2} \Big) ,$$

and also

$$\begin{aligned} A \cdot \mathbf{u}(t) &= e^{\lambda_1 t}\, A \cdot \mathbf{v_1} + e^{\lambda_2 t}\, A \cdot \mathbf{v_2} \\ &= \lambda_1 \Big(e^{\lambda_1 t} \mathbf{v_1} \Big) + \lambda_2 \Big(-2 e^{\lambda_2 t} \mathbf{v_2} \Big) \ , \end{aligned}$$

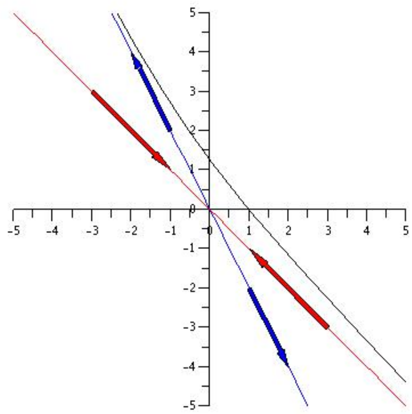
and thus

$$\dot{\mathbf{u}}(t) = A \cdot \mathbf{u}(t) \ ,$$

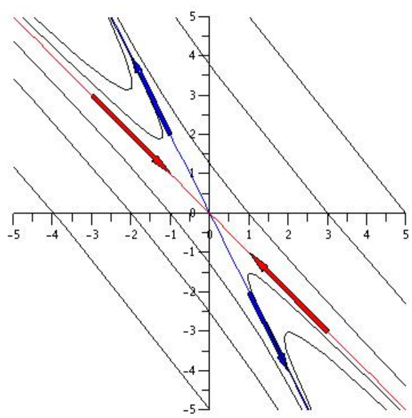
and $\mathbf{u}(t)$ is in fact the solution to the ODE we were looking for. In conclusion, we have

$$\mathbf{u}(t) = e^t \begin{bmatrix} -1 \\ 2 \end{bmatrix} - 2e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \ .$$

Here is the picture of the eigenlines plus the trajectory of $\mathbf{u}(t)$:



And here is the phase portrait:



5. Same sequence of steps for $\begin{cases} \dot{x} &= \quad 4x + 3y \\ \dot{y} &= -6x - 5y \end{cases}$

Ans.: We can write the equations for x, y as

$$\dot{\mathbf{u}}(t) \quad = \quad \underbrace{\begin{bmatrix} 4 & 3 \\ -6 & -5 \end{bmatrix}}_{=B} \mathbf{u}(t) \ .$$

We observe that $B = -2A^{-1}$. We compute

$$\operatorname{tr} B = -1 \,, \qquad \det B = -2 \,.$$

Thus, the characteristic polynomial is the same as for A, and so are the roots and eigenvalues. Thus $\lambda_1 = 1$ and $\lambda_2 = -2$.

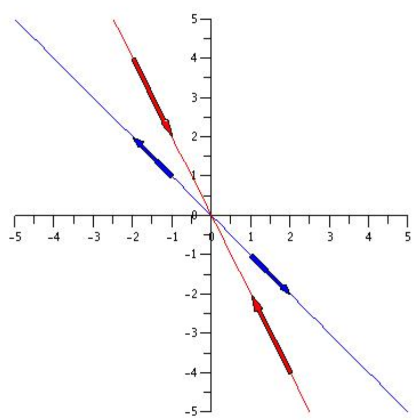
For $\lambda_1 = 1$, the eigenline is given by

$$\begin{bmatrix} -c \\ c \end{bmatrix}$$

with c a real number, but not zero. For $\lambda_2 = -2$ the eigenline is given by

$$\begin{bmatrix} -c' \\ 2c' \end{bmatrix}$$

with c' a real number but not zero. We see that all we have done is exchanging the roles of the eigenvectors $\mathbf{v_1}$ and $\mathbf{v_2}$ from the first example. Thus, we have the following picture for the eigenlines:



Now,

$$\mathbf{u}(t) = \alpha e^{t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \beta e^{-2t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

is a solution to the ODE with initial condition

$$\mathbf{u}(0) = \begin{bmatrix} -\alpha - \beta \\ \alpha + 2\beta \end{bmatrix} \,.$$

Thus, for $\alpha = -2$ and $\beta = 1$ it runs through the point $(1, 0)$. Here is the picture of the phase portrait: