18.03 Recitation 22, May 4, 2006

Complex or Repeated Eigenvalues

Solution suggestions

Find basic real solutions for $\dot{\mathbf{u}} = A\mathbf{u}$ with

1. $A = \begin{bmatrix} -6 & -8 \\ 5 & 6 \end{bmatrix}$.

Ans. Let's compute the characteristic polynomial of A. It's

$$\begin{aligned} p_A(\lambda) &= \det(A - \lambda I) \\ &= \lambda^2 - \mathrm{tr}\,(A)\,\lambda + \det(A) = \lambda^2 + 4\,. \end{aligned}$$

Thus, the roots are $\lambda = 2i$ and $\overline{\lambda} = -2i$. The characteristic polynomial has only real coefficients, thus the roots – if complex – must be complex conjugates of each other which is exactly what we have.

Now, we have to find the corresponding eigenvector but only for λ. We have to find $\mathbf{v}$ such that

$$A \cdot \mathbf{v} = \lambda\,\mathbf{v} = 2i\,\mathbf{v}\,.$$

If $\mathbf{v}$ has components a, b then the equation above can be written as

$$\begin{bmatrix} -6 & -8 \\ 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = 2i \begin{bmatrix} a \\ b \end{bmatrix}\,,$$

which becomes

$$\begin{aligned} -6a - 8b &= 2i\,a \\ 5a + 6b &= 2i\,b \end{aligned}$$

or

$$\begin{aligned} (-6-2i)a \;-\; 8b &= 0 \\ 5a \;+\; (6-2i)b &= 0 \end{aligned}\,.$$

If we mulitply the first equation with $\frac{5}{-6-2i}$ we get the second equation as

$$-8\,\frac{5}{(-6-2i)} = -8\,\frac{5(-6+2i)}{36+4} = 6 - 2i\,.$$

Thus, we get $-(6+2i)a = 8b$ or $-(3+i)a = 4b$ as our only equation. If we pick $b = 3+i$ we get $a = -4$. So, we obtain

$$\mathbf{v} = \begin{bmatrix} -4 \\ 3+i \end{bmatrix}\,.$$

This means that we have found the following complex eigenvalue, complex eigenvector, and complex solution to the ODE:

$$\begin{aligned} \lambda &= 2i\,, \qquad \mathbf{v} = \begin{bmatrix} -4 \\ 3+i \end{bmatrix}, \\ \mathbf{u}(t) &= e^{\lambda t}\,\mathbf{v} = e^{2i\,t}\begin{bmatrix} -4 \\ 3+i \end{bmatrix} = \Big(\cos(2t) + i\sin(2t)\Big)\left(\begin{bmatrix} -4 \\ 3 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right). \end{aligned}$$

But we wanted real solutions. As in the case of second order equations, the real and imaginary parts of solutions are again solutions. So these are real solutions:

$$\begin{aligned} \mathbf{u_1}(t) &= \operatorname{Re}\mathbf{u}(t) = \cos(2t)\begin{bmatrix} -4 \\ 3 \end{bmatrix} - \sin(2t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ \mathbf{u_2}(t) &= \operatorname{Im}\mathbf{u}(t) = \cos(2t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin(2t)\begin{bmatrix} -4 \\ 3 \end{bmatrix}. \end{aligned}$$

2. $A = \begin{bmatrix} -15 & -25 \\ 8 & 13 \end{bmatrix}$.

Ans. Let's compute the characteristic polynomial of A. It's

$$p_A(\lambda) = \lambda^2 - \operatorname{tr}(A)\,\lambda + \det(A) = \lambda^2 + 2\lambda + 5\,.$$

Thus, the roots are $\lambda = -1+2i$ and $\overline{\lambda} = -1-2i$.

Now, we have to find the corresponding eigenvector for λ. We have to find $\mathbf{v}$ such that

$$A \cdot \mathbf{v} = \lambda\,\mathbf{v} = (-1+2i)\,\mathbf{v}\,.$$

If $\mathbf{v}$ has components a, b then the equation above can be written as

$$\begin{bmatrix} -15 & -25 \\ 8 & 13 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = (-1+2i)\begin{bmatrix} a \\ b \end{bmatrix},$$

which becomes

$$\begin{aligned} (-14-2i)a \; - \; 25b &= 0 \\ 8a \; + \; (14-2i)b &= 0 \end{aligned}.$$

If we mulitply the first equation with $\frac{8}{-14-2i}$ we get the second equation as

$$-25\,\frac{8}{(-14-2i)} = -25\,\frac{4(-7+i)}{49+1} = 14-2i\,.$$

Thus, we get $8a = -(14-2i)b$ or $4a = (-7+i)b$ as our only equation. If we pick $b = 4$ we get $a = -7+i$. So, we obtain

$$\mathbf{v} = \begin{bmatrix} -7+i \\ 4 \end{bmatrix}.$$

This means that we have found the following complex eigenvalue, complex eigenvector, and complex solution to the ODE:

$$\begin{aligned} \lambda &= -1+2i\,, \qquad \mathbf{v} = \begin{bmatrix} -7+i \\ 4 \end{bmatrix}, \\ \mathbf{u}(t) &= e^{\lambda t}\,\mathbf{v} = e^{(-1+2i)\,t}\begin{bmatrix} -7+i \\ 4 \end{bmatrix} = \Big(\cos(2t) + i\sin(2t)\Big)\,e^{-t}\left(\begin{bmatrix} -7 \\ 4 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right). \end{aligned}$$

But we wanted real solutions. As in the case of second order equations, the real and imaginary parts of solutions are again solutions. So these are real solutions:

$$\begin{aligned} \mathbf{u_1}(t) &= \operatorname{Re}\mathbf{u}(t) = \cos(2t)\, e^{-t} \begin{bmatrix} -7 \\ 4 \end{bmatrix} - \sin(2t)\, e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \mathbf{u_2}(t) &= \operatorname{Im}\mathbf{u}(t) = \cos(2t)\, e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin(2t)\, e^{-t} \begin{bmatrix} -7 \\ 4 \end{bmatrix}. \end{aligned}$$

3. $A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$.

Ans. Let's compute the characteristic polynomial of A. It's

$$\begin{aligned} p_A(\lambda) &= \det(A - \lambda I) \\ &= \lambda^2 - \operatorname{tr}(A)\, \lambda + \det(A) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 . \end{aligned}$$

Thus, the (only repeated) root is $\lambda_1 = 1$.

First, we have to find the corresponding eigenvector for λ_1. We have to find $\mathbf{v}$ such that

$$A \cdot \mathbf{v} = \lambda_1 \mathbf{v} = \mathbf{v} .$$

If $\mathbf{v}$ has components a, b then the equation above can be written as

$$\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix},$$

which becomes

$$\begin{aligned} -a + b &= 0 \\ -a + b &= 0 \end{aligned} .$$

Thus, we get $a = b$ as our only equation. If we pick $b = 1$ we get $a = 1$. So, we obtain

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

This means that we have found the following (repeated) eigenvalue, eigenvector, and solution to the ODE:

$$\begin{aligned} \lambda_1 &= 1, \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ \mathbf{u}(t) &= e^{\lambda t}\, \mathbf{v} = e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} . \end{aligned}$$

But we need two real independent solutions. To do so, we have to find $\mathbf{w}$ such that

$$\begin{aligned} (A - \lambda_1 I)\mathbf{w} &= \mathbf{v} \\ \left(\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \mathbf{w} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} . \end{aligned}$$

If $\mathbf{w}$ has components c, d then the equation above can be written as

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

Thus, we get $d = 1 + c$ as our only equation. If we pick $c = 0$ we get $d = 1$. So, we obtain

$$\mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} .$$

Let's write down

$$\mathbf{w}(t) = e^{\lambda_1 t} (t\mathbf{v} + \mathbf{w}) = e^t \left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = e^t \begin{bmatrix} t \\ t+1 \end{bmatrix} .$$

We check that

$$\begin{aligned} \dot{\mathbf{w}}(t) &= \lambda_1 e^{\lambda_1 t} (t\mathbf{v} + \mathbf{w}) + e^{\lambda_1 t}\mathbf{v} \\ &= t\, e^{\lambda_1 t} \lambda_1 \mathbf{v} + e^{\lambda_1 t} (\mathbf{v} + \lambda_1 \mathbf{w}) \ . \end{aligned}$$

Now, by our construction we have

$$A\mathbf{v} = \lambda_1 \mathbf{v} \ , \qquad A\mathbf{w} = \mathbf{v} + \lambda_1 \mathbf{w} \ .$$

Therefore, we obtain

$$\begin{aligned} \dot{\mathbf{w}}(t) &= t\, e^{\lambda_1 t} \lambda_1 \mathbf{v} + e^{\lambda_1 t} (\mathbf{v} + \lambda_1 \mathbf{w}) \\ &= t\, e^{\lambda_1 t} A\mathbf{v} + e^{\lambda_1 t} A\mathbf{w} \\ &= A\, e^{\lambda_1 t} (t\mathbf{v} + \mathbf{w}) \\ &= A\, \mathbf{w}(t) \ . \end{aligned}$$

Thus, $\mathbf{w}(t)$ is a second real independent solution to the ODE.

4. Note that $A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$ is a companion matrix. What is the corresponding second order ODE? Find a basis for its solutions and compare them with what you found using matrix methods.

Ans. If $\mathbf{u}$(t) has components $x(t), y(t)$ then the ODE

$$\dot{\mathbf{u}}(t) = A\, \mathbf{u}(t)$$

can be written as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} ,$$

which becomes

$$\begin{aligned} \dot{x}(t) &= y(t) \\ \dot{y}(t) &= -x(t) + 2y(t) \ . \end{aligned}$$

If we use the first equation to eliminate $\dot{y}(t) = \ddot{x}(t)$ and $y(t) = \dot{x}(t)$ from the second equation we obtain

$$\ddot{x}(t) - 2\dot{x}(t) + x(t) = 0 \ .$$

We know how to write down two independent solutions to this ODE. The characteristic polynomial is

$$p(\lambda) = \lambda^2 - 2\lambda + 1$$

and agrees with $p_A(\lambda)$ above. Again, we have one repeated root, $\lambda_1 = 1$. The two indpendent solutions are

$$x_1(t) = e^{\lambda_1 t} = e^t \ , \qquad x_2(t) = te^{\lambda_1 t} = te^t \ .$$

These solutions are exactly the first components of $\mathbf{u}(\mathbf{t})$ and $\mathbf{w}(t)$.