18.03 Recitation 24, May 11, 2006

Matrix exponentials and inhomogeneous equations

A fundamental matrix for a square matrix A is a matrix of functions, $\Phi(t)$, whose columns are linearly independent solutions to $\dot{\mathbf{u}} = A\mathbf{u}$. The fundamental matrix with $\Phi(0) = I$ is the "matrix exponential" e^{At}. The solution to $\dot{\mathbf{u}} = A\mathbf{u}$ with initial condition $\mathbf{u}(0)$ is given by $e^{At}\mathbf{u}(0)$.

Variation of parameters: The general solution to the inhomogeneous equation $\dot{\mathbf{u}} = A\mathbf{u} + \mathbf{q}(t)$ is given by

$$\mathbf{u} = \Phi(t) \int \Phi(t)^{-1}\mathbf{q}(t)\, dt$$

where $\Phi(t)$ is any fundamental matrix. The contant of integration (a column vector $\mathbf{c}$) provides the general homogeneous solution $\Phi(t)\mathbf{c}$.

If $\mathbf{q}(t)$ is constant, and A is invertible, then $\mathbf{u_p}(t) = -A^{-1}\mathbf{q}$ is a solution.

These problems center on $A = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}$.

1. Find a fundamental matrix for A.

2. Find the exponential matrix e^{At}.

3. Find the solution to $\dot{\mathbf{u}} = A\mathbf{u}$ with $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

4. Find a solution to $\dot{\mathbf{u}} = A\mathbf{u} + \begin{bmatrix} 5 \\ 10 \end{bmatrix}$. What is the general solution?

5. Find a solution to $\dot{\mathbf{u}} = A\mathbf{u} + e^t \begin{bmatrix} 5 \\ 10 \end{bmatrix}$.