18.03 Recitation 24, May 11, 2006

Matrix exponentials and inhomogeneous equations

Solution suggestions

These problems center on $A = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}$.

1. Find a fundamental matrix for A.

Ans. First, we have to determine the eigenvalues and eigenvectors of A. The characteristic polynomial is

$$p_A(\lambda) = \lambda^2 - \mathrm{tr}\,(A)\lambda + \det(A) = \lambda^2 - 2\lambda + 5\ .$$

The roots of the characteristic polynomial are the eigenvalues. These are

$$1 \pm \sqrt{1-5} = 1 \pm 2i\ .$$

We have two complex eigenvalues which are complex conjugates of each other. We pick one, say $\lambda = 1 + 2i$ and determine its eigenvector $\mathbf{v}$. This means that we have to find $\mathbf{v}$ such that

$$A\mathbf{v} = \lambda\mathbf{v} = (1+2i)\mathbf{v}\ .$$

If $\mathbf{v}$ has components x and y this equation can be written out in components as

$$\begin{array}{rcccl} x & + & y & = & (1+2i)\ x\ , \\ -4\ x & + & y & = & (1+2i)\ y\ , \end{array}$$

or

$$\begin{array}{rcccl} -2i\ x & + & y & = & 0\ , \\ -4\ x & - & 2i\ y & = & 0\ . \end{array}$$

By multiplying the first equation with $-2i$ we obtain the second. Thus, $-2ix + y = 0$ or $y = 2ix$ is our only equation. If we pick $x = 1$ we get $y = 2i$, and thus

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2i \end{bmatrix}\ .$$

Thus we have found the complex solution

$$\mathbf{u}(t) = e^{\lambda t}\mathbf{v} = e^t\Big(\cos(2t) + i\sin(2t)\Big)\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)\ .$$

The real solutions $\mathbf{v_1}(t)$ and $\mathbf{v_2}(t)$ are given by the real and imaginary part of it. Thus, we find

$$\begin{aligned} \mathbf{v_1}(t) &= \mathrm{Re}\,\mathbf{u}(t) = e^t\cos(2t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} - e^t\sin(2t)\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} e^t\cos(2t) \\ -2e^t\sin(2t) \end{bmatrix}, \\ \mathbf{v_2}(t) &= \mathrm{Im}\,\mathbf{u}(t) = e^t\cos(2t)\begin{bmatrix} 0 \\ 2 \end{bmatrix} + e^t\sin(2t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^t\sin(2t) \\ 2e^t\cos(2t) \end{bmatrix}. \end{aligned}$$

The fundamental matrix $\mathbf{\Phi}(t)$ is the matrix with $\mathbf{v_1}(t)$ and $\mathbf{v_2}(t)$ as its columns, i.e.

$$\mathbf{\Phi}(t) = [\mathbf{v_1}(t)\ \mathbf{v_2}(t)] = \begin{bmatrix} e^t\cos(2t) & e^t\sin(2t) \\ -2e^t\sin(2t) & 2e^t\cos(2t) \end{bmatrix}.$$

2. Find the exponential matrix e^{At}.

Ans. From the lecture we know

$$e^{At} = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}.$$

We notice that

$$\mathbf{\Phi}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Remember that the inverse of a 2 by 2 matrix is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{1}$$

Thus, we have

$$\mathbf{\Phi}(0)^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

and

$$e^{At} = \begin{bmatrix} e^t\cos(2t) & e^t\sin(2t) \\ -2e^t\sin(2t) & 2e^t\cos(2t) \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} e^t\cos(2t) & \frac{1}{2}e^t\sin(2t) \\ -2e^t\sin(2t) & e^t\cos(2t) \end{bmatrix}.$$

3. Find the solution to $\dot{\mathbf{u}} = A\mathbf{u}$ with $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Ans. We write

$$\mathbf{u}(t) = e^{At}\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Taking the derivative we obtain

$$\dot{\mathbf{u}}(t) = A\,e^{At}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = A\mathbf{u}(t),$$

and plugging in zero we get

$$\mathbf{u}(0) = e^{A\,0}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = I\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Thus, $\mathbf{u}(t)$ is the desired solution. We compute

$$\mathbf{u}(t) = e^{At}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} e^t\cos(2t) & \frac{1}{2}e^t\sin(2t) \\ -2e^t\sin(2t) & e^t\cos(2t) \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} e^t(\cos(2t)+\sin(2t)) \\ 2e^t(\cos(2t)-\sin(2t)) \end{bmatrix}.$$

4. Find a solution to $\dot{\mathbf{u}} = A\mathbf{u} + \begin{bmatrix} 5 \\ 10 \end{bmatrix}$. What is the general solution?

Ans. Since

$$\mathbf{q} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

is constant, a particular solution is given by

$$\begin{aligned} \mathbf{u_p}(t) &= -A^{-1}\mathbf{q} = -\begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}^{-1} \mathbf{q} = -\frac{1}{\det(A)} \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \end{bmatrix} \\ &= -\frac{1}{5} \begin{bmatrix} -5 \\ 30 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}, \end{aligned}$$

where we have used Eq. (1) to compute the inverse of the matrix A.

The general solution is then given by

$$\mathbf{u}(t) = \mathbf{u_p}(t) + C_1\, \mathbf{v_1}(t) + C_2\, \mathbf{v_2}(t) \; .$$

The last two terms can be written in compact form with the help of the fundamental matrix. We obtain

$$\mathbf{u}(t) = \mathbf{u_p}(t) + \mathbf{\Phi}(t) \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \end{bmatrix} + \begin{bmatrix} e^t \cos(2t) & e^t \sin(2t) \\ -2e^t \sin(2t) & 2e^t \cos(2t) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} .$$

5. Find a solution to $\dot{\mathbf{u}} = A\mathbf{u} + e^t \begin{bmatrix} 5 \\ 10 \end{bmatrix}$.

Ans. A particular solution is given by

$$\mathbf{u_p}(t) = \mathbf{\Phi}(t) \int \mathbf{\Phi}(t)^{-1} e^t \begin{bmatrix} 5 \\ 10 \end{bmatrix} \, dt \; .$$

Because it is needed to compute the inverse, let's compute

$$\det \mathbf{\Phi}(\mathbf{t}) = \det \begin{bmatrix} e^t \cos(2t) & e^t \sin(2t) \\ -2e^t \sin(2t) & 2e^t \cos(2t) \end{bmatrix} = 2e^{2t} \cos^2(2t) + 2e^{2t} \sin^2(2t) = 2e^{2t} \; ,$$

where we have used the trigonometric identity $\cos^2(2t) + \sin^2(2t) = 1$. Now, we can compute the inverse using Eq. (1)

$$\mathbf{\Phi}(t)^{-1} = \frac{1}{2e^{2t}} \begin{bmatrix} 2e^t \cos(2t) & -e^t \sin(2t) \\ 2e^t \sin(2t) & e^t \cos(2t) \end{bmatrix} = \begin{bmatrix} e^{-t} \cos(2t) & -\frac{1}{2}e^{-t} \sin(2t) \\ e^{-t} \sin(2t) & \frac{1}{2}e^{-t} \cos(2t) \end{bmatrix} .$$

Thus, we have

$$\mathbf{\Phi}(t)^{-1} e^t \begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} \cos(2t) & -\frac{1}{2}\sin(2t) \\ \sin(2t) & \frac{1}{2}\cos(2t) \end{bmatrix} \begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 5(\cos(2t) - \sin(2t)) \\ 5(\cos(2t) + \sin(2t)) \end{bmatrix} .$$

Integrating yields

$$\int \mathbf{\Phi}(t)^{-1} e^t \begin{bmatrix} 5 \\ 10 \end{bmatrix} \, dt = \begin{bmatrix} 5\int(\cos(2t) - \sin(2t))dt \\ 5\int(\cos(2t) + \sin(2t))dt \end{bmatrix} = \frac{5}{2} \begin{bmatrix} \sin(2t) + \cos(2t) \\ \sin(2t) - \cos(2t) \end{bmatrix} .$$

Finally, we have to multiply the fundamental matrix with this vector

$$\begin{aligned}
\mathbf{u_p}(t) &= \frac{5}{2}\mathbf{\Phi}(t)\begin{bmatrix} \sin(2t)+\cos(2t) \\ \sin(2t)-\cos(2t) \end{bmatrix} \\
&= \frac{5}{2}\begin{bmatrix} e^t\cos(2t) & e^t\sin(2t) \\ -2e^t\sin(2t) & 2e^t\cos(2t) \end{bmatrix}\begin{bmatrix} \sin(2t)+\cos(2t) \\ \sin(2t)-\cos(2t) \end{bmatrix} \\
&= \frac{5e^t}{2}\begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}\begin{bmatrix} \sin(2t)+\cos(2t) \\ \sin(2t)-\cos(2t) \end{bmatrix}.
\end{aligned}$$

If we substitute $c=\cos(2t)$ and $s=\sin(2t)$, this can be written as

$$\frac{5e^t}{2}\begin{bmatrix} c & s \\ -2s & 2c \end{bmatrix}\begin{bmatrix} s+c \\ s-c \end{bmatrix} = \frac{5e^t}{2}\begin{bmatrix} c(s+c)+s(s-c) \\ -2s(s+c)+2c(s-c) \end{bmatrix} = \frac{5e^t}{2}\begin{bmatrix} c^2+s^2 \\ -2s^2-2c^2 \end{bmatrix} = \frac{5e^t}{2}\begin{bmatrix} 1 \\ -2 \end{bmatrix},$$

where we have used the trigonometric identity $\cos^2(2t)+\sin^2(2t)=1$.

Thus, a particular solution is given by

$$\mathbf{u_p}(t) = \begin{bmatrix} \frac{5e^t}{2} \\ -5e^t \end{bmatrix}.$$