18.03 Recitation 25, May 16, 2006

Qualitative analysis of nonlinear systems

Solution suggestions

1. (a) Talk through an interpretation of the nonlinear autonomous system

$$\dot{x} = \underbrace{(6 - x - 2y)x}_{f(x,y)}$$

$$\dot{y} = \underbrace{(6 - y - 2x)y}_{g(x,y)}$$

in terms of two populations. What effect do they have on each other's growth rates?

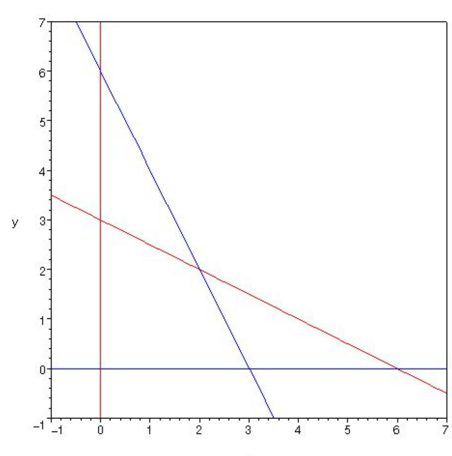
Ans. The autonomous system above modells the following situation: we have two species that compete with each other for the same food. Each population depresses the growth rate of the other, as well as its own (logistically).

(b) Find where this vector field is vertical and where it is horizontal and plot these curves (or lines). Find the critical points. Compute the Jacobian matrix

$$J(x,y) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix}$$

For each critical point (a,b), compute $J(a,b)$. Is the critical point nondegenerate? If so, what is the linearization at it? If node or saddle, determine the eigenlines with their arrows of time, and if node, which eigenline solutions become tangent to as the near the critical point. Snap these little sketches in and complete to a phase portrait.

Ans. The vector field is vertical if $\dot{x} = 0$ (red lines), i.e. if $x = 0$ (this is the y-axis) or $6 - x - 2y = 0$ (this is the line $y = -\frac{1}{2}x + 3$). Similarly, the vector field is horizontal if $\dot{y} = 0$ (blue lines), i.e. $y = 0$ (this is the x-axis) or $6 - y - 2x = 0$ (this is the line $y = -2x + 6$). Here is the picture of these lines:



The critical points are the intersection points of a blue line with a red line. Thus, the critical points are $(0,0), (6,0), (0,6), (2,2)$. We compute the Jacobian

$$J(x,y) = \begin{bmatrix} 6-2x-2y & -2x \\ -2y & 6-2y-2x \end{bmatrix} .$$

At a critical point (x_0, y_0) we have $f(x_0, y_0) = 0$ and $g(x_0, y_0) = 0$ (this was exactly the condition that a red and a blue line intersect). If the matrix $J(x_0, y_0)$ is not zero we can approximate the autonomous system

$$\dot{x} = \underbrace{(6-x-2y)x}_{f(x,y)}$$

$$\dot{y} = \underbrace{(6-y-2x)y}_{g(x,y)}$$

by

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0 \end{bmatrix} + J(x_0, y_0) \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} .$$

Now, if we introduce $a = x - x_0$ and $b = y - y_0$, we have $\dot{a} = \dot{x}$ and $\dot{b} = \dot{y}$ and thus

$$\begin{bmatrix} \dot{a} \\ \dot{b} \end{bmatrix} \approx J(x_0, y_0) \begin{bmatrix} a \\ b \end{bmatrix} .$$

This is a linear system (for each critical point (x_0, y_0)). Let's look at the different critical points:

at (0,0): at the critical point $(0,0)$ the Jacobian is

$$J(0,0) = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} .$$

This matrix is already diagonal. This means we have the repeated eigenvalue 6. The eigenvectors are $[1;0]$ and $[0;1]$. The general solution is

$$c_1 e^{6t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{6t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} .$$

This is an unstable star. This means that around $(0,0)$ we have flow-lines running in in the shape of a star – since both eigenvectors have the same eigenvalue.

at (0,6): at the critical point $(0,6)$ the Jacobian is

$$J(0,6) = \begin{bmatrix} -6 & 0 \\ -12 & -6 \end{bmatrix} .$$

The characteristic polynomial is

$$p_{(0,6)}(\lambda) = \lambda^2 + 12\lambda + 36 = (\lambda+6)^2 .$$

Thus, the only eigenvalue is $\lambda = -6$. And we find only *one* eigenvector

$$\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} .$$

So, this is the case of a stable, defective node.

To write down the general solution we have to find a vector $\mathbf{w}$ such that

$$\Big(J(0,6) - \lambda I\Big)\mathbf{w} = \begin{bmatrix} 0 & 0 \\ -12 & 0 \end{bmatrix} \mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} .$$

We find

$$\mathbf{w} = \begin{bmatrix} -\frac{1}{12} \\ 0 \end{bmatrix} .$$

The general solution is then given by

$$e^{-6t}\left\{ c_1 \left(\begin{bmatrix} 0 \\ t \end{bmatrix} + \begin{bmatrix} -\frac{1}{12} \\ 0 \end{bmatrix} \right) + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = e^{-6t} \begin{bmatrix} -\frac{c_1}{12} \\ c_1 t + c_2 \end{bmatrix} .$$

Around $(0,6)$ we have flow lines running in, in the form of a defective node. The trajectories corresponding to $c_1 = 0$ start above and below of the critical point on the y-axis and run straight towards the critical point on the y-axis.

at (6,0): at the critical point $(6,0)$ the Jacobian is

$$J(6,0) = \begin{bmatrix} -6 & -12 \\ 0 & -6 \end{bmatrix} .$$

The characteristic polynomial is

$$p_{(6,0)}(\lambda) = \lambda^2 + 12\lambda + 36 = (\lambda + 6)^2 .$$

Thus, the only eigenvalue is again $\lambda = -6$. And we find only *one* eigenvector

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} .$$

So, this is again the case of a stable, defective node. To write down the general solution we have to find a vector $\mathbf{w}$ such that

$$\Big(J(6,0) - \lambda I\Big)\mathbf{w} = \begin{bmatrix} 0 & -12 \\ 0 & 0 \end{bmatrix} \mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} .$$

We find

$$\mathbf{w} = \begin{bmatrix} 0 \\ -\frac{1}{12} \end{bmatrix} .$$

The general solution is then given by

$$e^{-6t}\left\{ c_1 \left(\begin{bmatrix} t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{12} \end{bmatrix} \right) + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} .$$

Around $(6,0)$ we have flow lines running in in the form of a defective node. The trajectories corresponding to $c_1 = 0$ start to the right and to the left of the critical point on the x-axis and run straight towards the critical point on the x-axis.

at (2,2): at the critical point $(2,2)$ the Jacobian is

$$J(2,2) = \begin{bmatrix} -2 & -4 \\ -4 & -2 \end{bmatrix} .$$

The characteristic polynomial is

$$p_{(2,2)}(\lambda) = \lambda^2 + 4\lambda - 12 \ .$$

Thus, the eigenvalues are $\lambda_1 = -6$ and $\lambda_2 = 2$. This is an unstable saddle. For the eigenvalue $\lambda_1 = -6$ we find the eigenvector

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} ,$$

and for the eigenvalue $\lambda_1 = 2$ we find the eigenvector

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} ,$$

The general solution is then given by

$$c_1 e^{-6t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} .$$

This means that around $(2, 2)$ trajectories that start at points on the eigenline with $c_1 \neq 0$ and $c_2 = 0$ (this is the line $y = x$) are running towards the critical point. All other trajectories with $c_2 \neq 0$ are flowing away from the critical point. We have

$$c_1 e^{-6t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \simeq_{t \to \infty} c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

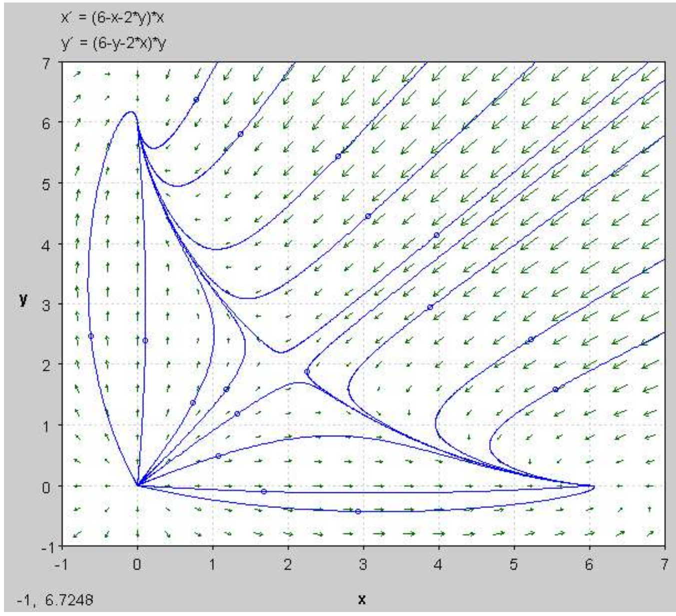
This means that all other trajectories that run through points not exactly on the line $y = x$ will converge to the line $y = -x$ (but remember we have to move this line to the critical point at $(2, 2)$ so the line really is $y - 2 = -(x - 2)$ or $y = -x + 4$). On this line, they will be running away from the critical point either up left or down and to the right.

Notice: our linearization around the various critical points might not be structurally stable. This means that the non-linear system around these point might be *not* stable and/or *not* defective/star even though the linear system is. But we do know that whether it has a node and also the tangent eigenline solutions will be the same as in the linear case. This allows us to complete a phase portrait.

A Java applet of the MATLAB's `pplane6` is available under:

`http://math.rice.edu/ dfield/dfpp.html`

Here is the phase portrait:



(c) Finally, pick an initial condition and sketch graphs of the two functions $x(t)$ and $y(t)$. You won't be able to be very precise about the time direction, given the information presented by the phase portrait. When t is large, however, you should be able to write down what $x(t)$ and $y(t)$ are to a good approximation, using the linearization.

Ans. We observe that if we start below the graph of $y = x$, the trajectory will approach the point $(6, 0)$. This means that asymptotically $x(t)$ becomes 6 while $y(t)$ will tend to zero. This means that the species described by y will be extincted. On the other hand, starting from above the $y = x$ line means that the solution will run towards the point $(0, 6)$. This means $x(t)$ will go to zero (extinction of the first species) while $y(t)$ will go to 6.

2. Find the general solution to the differential equation
$\dot{\mathbf{u}} = \underbrace{\begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix}}_{A} \mathbf{u} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Sketch the phase portrait.

Ans. Since $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is time independent a particular solution is given by

$$\mathbf{u}_p(t) = -A^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -\frac{1}{\det A} \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -\frac{1}{4} \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} .$$

To get the general solution we have to add to this the general solution of the homogeneous system

$$\dot{\mathbf{u}} = A\mathbf{u} .$$

The characteristic polynomial is

$$p_A(\lambda) = \lambda^2 - \operatorname{tr}(A)\,\lambda + \det(A) = \lambda^2 + 4 .$$

The roots are $\pm 2i$. This is the case called "center". We pick the eigenvalue $\lambda = 2i$, and compute the eigenvector $\mathbf{v}$. This means we have to find $\mathbf{v}$ such that

$$A\mathbf{v} = \lambda\,\mathbf{v} = 2i\,\mathbf{v} .$$

One find

$$\mathbf{v} = \begin{bmatrix} 1 \\ -2i \end{bmatrix}.$$

This means that we have a complex solution of the form

$$e^{\lambda t}\mathbf{v} = \Big(\cos(2t) + i\sin(2t)\Big)\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ -2 \end{bmatrix}\right).$$

The real and the imaginary part are two independent solutions to the homogeneous ODE. We compute

$$\begin{aligned} \operatorname{Re} e^{\lambda t}\mathbf{v} &= \begin{bmatrix} \cos(2t) \\ 2\sin(2t) \end{bmatrix}, \\ \operatorname{Im} e^{\lambda t}\mathbf{v} &= \begin{bmatrix} \sin(2t) \\ -2\cos(2t) \end{bmatrix}. \end{aligned}$$

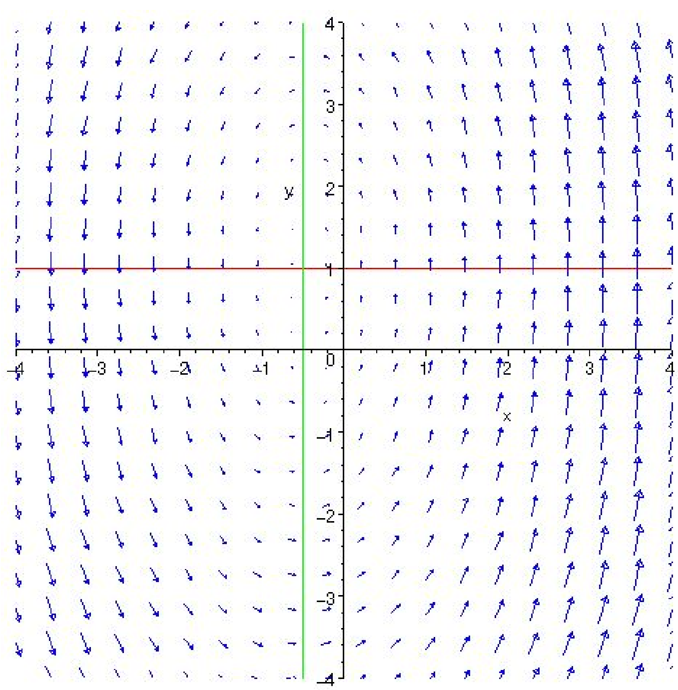
Thus, the general solution to the original ODE is given by

$$\begin{aligned} \mathbf{u}(t) &= \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + c_1\begin{bmatrix} \cos(2t) \\ 2\sin(2t) \end{bmatrix} + c_2\begin{bmatrix} \sin(2t) \\ -2\cos(2t) \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + \underbrace{\begin{bmatrix} \cos(2t) & \sin(2t) \\ 2\sin(2t) & -2\cos(2t) \end{bmatrix}}_{\Phi(t)}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \end{aligned}$$

If $\mathbf{u}$ has components x and y we have to sketch

$$\dot{\mathbf{u}}(t) = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -y+1 \\ 4x+2 \end{bmatrix}$$

This means for each point with coordinates (x, y) in the plane we draw a little vector starting at the point (x, y) and pointing in the direction $[\dot{x}; \dot{y}] = [-y+1; 4x+2]$. It is horizontal, i.e. $\dot{y} = 0$, along $x = -1/2$ (green line) and vertical, i.e. $\dot{x} = 0$, along $y = 1$ (red line). The picture looks like this:



3. Now use variation of parameters to find a particular solution to
$\dot{\mathbf{u}} = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix} \mathbf{u} + \begin{bmatrix} t \\ 0 \end{bmatrix}$.
Is there a phase portrait for this equation?

Ans. By the variation of constants the solution is given by

$$\mathbf{u}(t) = \Phi(t) \int \Phi^{-1}(t) \begin{bmatrix} t \\ 0 \end{bmatrix} \, dt \, .$$

We compute

$$\det \Phi(t) = -2\cos^2(2t) - 2\sin^2(2t) = -2 \, ,$$

and thus

$$\Phi(t)^{-1} = \begin{bmatrix} \cos(2t) & \sin(2t) \\ 2\sin(2t) & -2\cos(2t) \end{bmatrix}^{-1} = \frac{1}{-2} \begin{bmatrix} -2\cos(2t) & -\sin(2t) \\ -2\sin(2t) & \cos(2t) \end{bmatrix} \, .$$

We obtain

$$\Phi(t)^{-1} \begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} t\cos(2t) \\ t\sin(2t) \end{bmatrix} \, .$$

Remember that integration by parts gives

$$\int t \, \cos(2t) \, dt = \frac{1}{4} \cos(2t) + \frac{1}{2} t \sin t + c_1 \, ,$$

and similarly for $t \, \sin(2t)$. Thus, integration yields

$$\int \Phi(t)^{-1} \begin{bmatrix} t \\ 0 \end{bmatrix} \, dt = \begin{bmatrix} \frac{1}{4}\cos(2t) + \frac{1}{2} t \sin t + c_1 \\ \frac{1}{4}\sin(2t) - \frac{1}{2} t \cos t + c_2 \end{bmatrix} \, .$$

Therefore, we obtain

$$\Phi(t) \int \Phi(t)^{-1} \begin{bmatrix} t \\ 0 \end{bmatrix} \, dt = \Phi(t) \begin{bmatrix} \frac{1}{4}\cos(2t) + \frac{1}{2} t \sin t \\ \frac{1}{4}\sin(2t) - \frac{1}{2} t \cos t \end{bmatrix} + \Phi(t) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \, .$$

The second term on the RHS just adds mutiples of the general solution of the homogeneous ODE. We compute

$$\Phi(t) \begin{bmatrix} \frac{1}{4}\cos(2t) + \frac{1}{2} t \sin t \\ \frac{1}{4}\sin(2t) - \frac{1}{2} t \cos t \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ t \end{bmatrix} \, .$$

If $\mathbf{u}$ has components x and y we had to sketch

$$\dot{\mathbf{u}}(t) = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} -y + t \\ 4x \end{bmatrix}$$

This means for each point with coordinates (x, y) in the plane we had to draw a little vector starting at the point (x, y) and pointing in the direction $[\dot{x}; \dot{y}] = [-y + t; 4x]$. We see that the direction and length of this vector is time-dependent. Therefore, we cannot draw a phase portrait as in **(2)**.

4. We saw how second order equations give rise to a "companion" first order system of equations, by means of the equation $y = \dot{x}$. Suppose $x(t) = 1 + \sin(2t)$. Sketch the trajectory of the vector-valued function $\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$.

Ans. For $x(t) = 1 + \sin(2t)$ we compute

$$y = \dot{x} = 2\cos(2t) \ ,$$

and therefore

$$\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \sin(2t) \\ 2\cos(2t) \end{bmatrix} \ .$$

The trajectory is an ellipse centered at the point $(1, 0)$. It starts at $t = 0$ at the point $(2, 2)$ and runs clockwise and returns to the starting point at $t = \pi$.