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18.03 Differential Equations, Spring 2006
Transcript – Lecture 4

The topic for today is how to change variables. So, we're talking about substitutions and differential equations, or changing variables. That might seem like a sort of fussy thing to talk about in the third or fourth lecture, but the reason is that so far, you know how to solve two kinds of differential equations, two kinds of first-order differential equations, one where you can separate variables, and the linear equation that we talked about last time.

Now, the sad fact is that in some sense, those are the only two general methods there are, that those are the only two kinds of equations that can always be solved. Well, what about all the others? The answer is that to a great extent, all the other equations that can be solved, the solution can be done by changing the variables in the equation to reduce it to one of the cases that we can already do. Now, I'm going to give you two examples of that, two significant examples of that today. But, ultimately, as you will see, the way the equations are solved is by changing them into a linear equation, or an equation where the variables are separable.

However, that's for a few minutes. The first change of variables that I want to talk about is an almost trivial one. But it's the most common kind there is, and you've already had it in physics class. But I think it's so important in the science and engineering subjects that it's a good idea, even in 18.03, to call attention to it explicitly. So, in that sense, the most common change of variables is the one simple one called scaling. So, again, the kind of equation I'm talking about is a general first-order equation. And, scaling simply means to change the coordinates, in effect, or axes, to change the coordinates on the axes to scale the axes, to either stretch them or contract them.

So, what does the change of variable actually look like? Well, it means you introduce new variables, where x1 is equal to x times something or times a constant. I'll write it as divided by a constant, since that tends to be a little bit more the way people think of it. And y, the same. So, the new variable y1 is related to the old one by an equation of that form. So, a, b are constants. So, those are the equations. Why does one do this? Well, for a lot of reasons. But, maybe we can list them. You, for example, could be changing units. That's a common reason in physics. Changing the units that he used, you would have to make a change of coordinates of this form.

Perhaps the even more important reason is to, sometimes it's used to make the variables dimensionless. In other words, so that the variables become pure numbers, with no units attached to them. Since you are well aware of the tortures involved in dealing with units in physics, the point of making variables, I'm sorry, dimensionless, I don't have to sell that. Dimensionless, i.e. no units, without any units attached. It just represents the number three, not three seconds, or three grahams, or anything like that.

And, the third reason is to reduce or simplify the constants: reduce the number or simplify the constants in the equation. Reduce their number is self explanatory.

Simplify means make them less, either dimensionless also, or if you can't do that, at least less dependent upon the critical units than the old ones were. Let me give you a very simple example which will illustrate most of these things. It's the equation. It's a version of the cooling law, which applies at very high temperatures, and it runs. So, it's like Newton's cooling laws, except it's the internal and external temperatures vary, what's important is not the first power as in Newton's Law, but the fourth power.

So, it's a constant. And, the difference is, now, it's the external temperature, which, just so there won't be so many capital T's in the equation, I'm going to call M, dT / dt = k(M^4 - T^4). So, T is the internal temperature, the thing we are interested in. And, M is the external constant, which I'll assume, now, is a constant external temperature.

So, this is valid if big temperature differences, Newton's Law, breaks down and one needs a different one. Now, you are free to solve that equation just as it stands, if you can. There are difficulties connected with it because you're dealing with the fourth powers, of course. But, before you do that, one should scale. How should I scale? Well, I'm going to scale by relating T to M. So, that is very likely to use is T1 = T / M. This is now dimensionless because M, of course, has the units of temperature, degrees Celsius, degrees absolute, whatever it is, as does T.

And therefore, by taking the ratio of the two, there are no units attached to it. So, this is dimensionless. Now, how actually do I change the variable in the equation? Well, watch this. It's an utterly trivial idea, and utterly important. Don't slog around doing it this way, trying to stuff it in, and divide first. Instead, do the inverse. In other words, write it instead as T equals MT1, the reason being that it's T that's facing you in that equation, and therefore T you want to substitute for. So, let's do it. The new equation will be what? Well, dT-- Since this is a constant, the left-hand side becomes dT1 / dt * M = k (M^4 - M^4 T1^4), so I'm going to factor out that M to the forth, and make it (1 - T1^4), okay?

Now, I could divide through by M and get rid of one of those, and so, the new equation, now, is dT1 / dt, d time, is equal to-- Now, I have k M^3 out front here. I'm going to just give that a new name, k1. Essentially, it's the same equation. It's no harder to solve and no easier to solve than the original one. But it's been simplified. For one, I think it looks better. So, to compare the two, I'll put this one up in green, and this one in green, too, just to convince you it's the same, but indicate that it's the same equation.

Notice, so, T1 has been rendered, is now dimensionless. So, I don't have to even ask when I solve this equation, oh, please tell me what the units of temperature are. How you are measuring temperature makes no difference to this equation. k1 still has units. What units does it have? It's been simplified because it now has the units of, since this is dimensionless and this is dimensionless, it has the units of inverse time. So, k1, whereas it had units involving both degrees and seconds before, now it has inverse time as its units. And, moreover, there's one less constant. So, one less constant in the equation.

It just looks better. This business, I think you know that k1, the process of forming k1 out of k M cubed is called lumping constants. I think they use standard terminology in physics and engineering courses. Try to get all the constants together like this. And then you lump them there. They are lumped for you, and then you just give the lump a new name. So, that's an example of scaling. Watch out for when you can use that. For example, it would have probably been a good thing to use in the first problem set when you were handling this problem of drug elimination and hormone elimination production inside of the thing. You could lump constants, and as was done to some extent on the solutions to get a neater looking answer, one without so many constants in it.

Okay, let's now go to serious stuff, where we are actually going to make changes of variables which we hope will render unsolvable equations suddenly solvable. Now, I'm going to do that by making substitutions. But, it's, I think, quite important to watch up there are two kinds of substitutions. There are direct substitutions. That's where you introduce a new variable. I don't know how to write this on the board. I'll just write it schematically. So, it's one which says that the new variable is equal to some combination of the old variables. The other kind of substitution is inverse. It's just the reverse.

Here, you say that the old variables are some combination of the new. Now, often you'll have to stick in a few old variables, too. But the basic, it's what appears on the left-hand side. Is it a new variable that appears on the left-hand side by itself, or is it the old variable that appears on the left-hand side? Now, right here, we have an example. If I did it as a direct substitution, I would have written $T1 = T / M$.

That's the way I define the new variable, which of course you have to do if you're introducing it. But when I actually did the substitution, I did the inverse substitution. Namely, I used $T = M*T1$. And, the reason for doing that was because it was the capital T's that faced me in the equation and I had to have something to replace them with. Now, you see this already in calculus, this distinction. But that might have been a year and a half ago. Just let me remind you, typically in calculus, for example, when you want to do this kind of integral, let's say, integral of $x*\mathrm{sqrt}(1 - x^2)\,dx$, the substitution you'd use for that is $u = 1 - x^2$, right? And then, you calculate, and then you would observe that this, the x dx, more or less makes up du, apart from a constant factor, there.

So, this would be an example of direct substitution. You put it in and convert the integral into an integral of u. What would be an example of inverse substitution? Well, if I take away the x and ask you, instead, to do this integral, then you know that the right thing to do is not to start with u, but to start with the x and write x equals sine or cosine u. So, this is a direct substitution in that integral, but this integral calls for an inverse substitution in order to be able to do it. And notice, they would look practically the same. But, of course, as you know from your experience, they are not. They're very different. Okay, so I'm going to watch for that distinction as I do these examples. The first one I want to do is an example as a direct substitution.

So, it applies to the equation of the form y prime equals, there are two kinds of terms on the right-hand side. Let's use p of x, $p(x)*y + q(x)*y^n$. Well, notice, for example, if n were zero, what kind of equation would this be? y^n would be one, and this would be a linear equation, which you know how to solve. So, n equals zero we already know how to do. So, let's assume that n is not zero, so that we're in new territory.

Well, if n were equal to one, you could separate variables. So, that too is not exciting. But, nonetheless, it will be included in what I'm going to say now. If n is two or three, or n could be one half. So anything: even zero is all right. It's just silly. Any number: it could be negative. n equals minus five. That would be fine also. This kind of equation, to give it its name, is called the Bernoulli equation, named after which Bernoulli, I haven't the faintest idea. There were, I think, three or four of them. And, they fought with each other. But, they were all smart. Now, the key trick, if you like, method, to solving any Bernoulli equation, let me call another thing. Most important is what's missing.

It must not have a pure x term in it. And that goes for a constant term. In other words, it must look exactly like this. Everything multiplied by y, or a power of y, two terms. So, for example, if I add one to this, the equation becomes non-doable. Right, it's very easy to contaminate it into an equation that's unsolvable. It's got to look just like that. Now, you've got one on your homework. You've got several. Both part one and part two have Bernoulli equations on them. So, this is practical, in some sense. What do we got? The idea is to divide by y^n. Ignore all formulas that you're given. Just remember that when you see something that looks like this, or something that you can turn into something that looks like this, divide through by y to the nth power, no matter what n is.

All right, so $y' / y^n = p(x)*1 / (y^{(n-1)}) + q(x)$. Well, that certainly doesn't look any better than what I started with. And, in your terms, it probably looks somewhat worse because it's got all those Y's at the denominator, and who wants to see them there? But, look at it. In this very slightly transformed Bernoulli equation is a linear equation struggling to be free. Where is it? Why is it trying to be a linear equation? Make a new variable, call this hunk of it in new variable. Let's call it V. So, $V = 1 / y^{(n-1)}$. Or, if you like, you can think of that as $y^{(1-n)}$.

What's V prime? So, this is the direct substitution I am going to use, but of course, the problem is, what am I going to use on this? Well, the little miracle happens. What's the derivative of this? It is $y' = (1-n)*y^{(-n)}*y'$. In other words, up to a constant, this constant factor, one minus n, it's exactly the left-hand side of the equation. Well, let's make N not equal one either. As I said, you could separate variables if n equals one. What's the equation, then, turned into?

A Bernoulli equation, divided through in this way, is then turned into the equation one minus n, sorry, $V' / (1 - n) = p(x)*V + q(x)$. It's linear. And now, solve it as a linear equation. Solve it as a linear equation. You notice, it's not in standard form, not in standard linear form. To do that, you're going to have to put the p on the other side. That's okay, that term, on the other side, solve it, and at the end, don't forget that you put in the V. It wasn't in the original problem. So, you have to convert the problem, the answer, back in terms of y. It'll come out in terms of V, but you must put it back in terms of y.

Let's do a really simple example just to illustrate the method, and to illustrate the fact that I don't want you to memorize formulas. Learn methods, not final formulas. So, suppose the equation is, let's say, $y' = y / x - y^2$. That's a Bernoulli equation. I could, of course, have concealed it by writing xy prime plus xy prime minus xy equals negative y squared. Then, it wouldn't look instantly like a Bernoulli equation. You would have to stare at it a while and say, hey, that's a Bernoulli equation. Okay, but so I'm handing it to you a silver platter, as it were. So, what do we do?

Divide through by y squared. So, it's $y' / y^2 = 1 / x * 1 / y - 1$. And now, the substitution, then, I'm going to make, is for this thing. $V = 1 / y$. It's a direct substitution. $V' = -1 / y^2 * y'$. Don't forget to use the chain rule when you differentiate with respect-- because the differentiation is with respect to x, of course, not with respect to y. Okay, so what's this thing? That's the left-hand side. The only thing is it's got a negative sign. So, this is minus V prime equals, one over x stays one over x, one over y. So, it's $V / x - 1$.

So, let's put that in standard form. In standard form, it will look like, first imagine multiplying it through by negative one, and then putting this term on the other side. And, it will turn into $V' + V / x = 1$. So, that's the linear equation in standard linear form that we are asked to solve. And, the solution isn't very hard. The integrating factor is, well, I integrate one over x.

That makes log x. And, e to the log x, so, it's e to the log x, which is, of course, just x itself. So, I should multiply this through by x squared, be able to integrate it. Now, some of you, I would hope, just can see that right away, that if you multiply this through by x, it's going to look good. So, after we multiply through by x, which I get? (xV) prime for the-- maybe I shouldn't skip a step.

You are still learning this stuff, so let's not skip a step. So, it becomes $xV' + V = x$, okay? After I multiplied through by the integrating factor, this now says this is xV prime, and I quickly check that that, in fact, is what it's equal to, equals x, and therefore $xV = 1/2\ x^2 + c$. And, therefore, $V = 1/2x + c / x$.

You can leave it at that form, or you can combine terms. It doesn't matter much. Am I done? The answer is, no I am not done, because nobody reading this answer would know what V was. V wasn't in the original problem. It was y that was in the original problem. And therefore, the relation is, one is the reciprocal of the other. And therefore, I have to turn this expression upside down. Well, if you're going to have to turn it upside down, you probably should make it look a little better. Let's rewrite it as $x^2 + 2c$, combining fractions, I think they call it in high school or elementary school, plus 2c.

How's that? $(x^2 + 2c) / (2x)$. Now, 2c, you don't call it constant 2c because this is just as arbitrary to call it c1. So, I'll call that, so, my answer will be $y = 2x / (x^2 + C1)$, where C1 is an arbitrary constant. To indicate it's different from that one, I'll call it C1. C1 is two times the old one, but that doesn't really matter. So, there's the solution.

It has an arbitrary constant in it, but you note it's not an additive arbitrary constant. The arbitrary constant is tucked into the solution. If you had to satisfy an initial condition, you would take this form, and starting from this form, figure out what C1 was in order to satisfy that initial condition. Thus, Bernoulli equation is solved. Our first Bernoulli equation: isn't that exciting? So, here was the equation, and there is its solution. Now, the one I'm asking you to solve on the problem set in part two is no harder than this, except I ask you some hard questions about it, not very hard, but a little hard about it.

I hope you will find them interesting questions. You already have the experimental evidence from the first problem set, and the problem is to explain the experimental evidence by actually solving the equation in the scene. I think you'll find it interesting. But, maybe that's just a pious hope. Okay, I like, now, to turn to the second method, where a second class of equations which require inverse substitution, and those are equations, which are called homogeneous, a highly overworked word in differential equations, and in mathematics in general.

But, it's unfortunately just the right word to describe them. So, these are homogeneous, first-order ODE's. Now, I already used the word in one context in talking about the linear equations when zero is the right hand side. This is different, but nonetheless, the two uses of the word have the same common source. The homogeneous differential equation, homogeneous newspeak, is y prime equals, it's a question of what the right hand side looks like. And, now, the supposed way to say it is, you should be able to write the right-hand side as a function of a combined variable, $F(y/x)$.

In other words, after fooling around with the right hand side a little bit, you should be able to write it so that every time a variable appears, it's always in the combination y/x. Let me give some examples. For example, suppose $y' = x^2 y / (x^3 + y^3)$. Well, that doesn't look in that form. Well, yes it is. Imagine dividing the top and bottom by x cubed. What would you get? The top would be y/x, if you divided it by x cubed. And, if I divide the bottom by x cubed, also, which, of course, doesn't change the value of the fraction, as they say in elementary school, $1 + (y/x)^3$. So, this is the way it started out looking, but you just said ah-ha, that was a homogeneous equation because I could see it could be written that way.

How about another homogeneous equation? How about x y'? Is that a homogeneous equation? Of course it is: otherwise, why would I be talking about it? If you divide through by x, you can tuck it inside the radical, the square root, if you remember to square it when you do that. And, it becomes $\sqrt{1 + y^2/x^2}$. It's homogeneous. Now, you might say, hey, this looks like you had to be rather clever to figure out if an equation is homogeneous. Is there some other way? Yeah, there is another way, and it's the other way which explains why it's called homogeneous. You can think of it this way.

It's an equation which is, in modern speak, invariant, invariant under the operation zoom. What is zoom? Zoom is, you increase the scale equally on both axes. So, the zoom operation is the one which sends x into a times x, and y into a times y. In other words, you change the scale on both axes by the same factor, a. Now, what I say is, gee, maybe you shouldn't write it like this. Why don't we say, we introduce, how about this?

So, think of it as a change of variables. We will write it like that. So, you can put here an equals sign, if you don't know what this meaningless arrow means. So, I'm making this change of variables, and I'm describing it as an inverse substitution. But of course, it wouldn't make any difference. It's exactly the same as the direct substitution I started out with underscaling. The only difference is, I'm not using different scales on both axes. I'm expanding them both equally.

That's what I mean by zoom. Now, what happens to the equation? Well, what happens to dy/dx? Well, $dx = (a)dx_1$. $dy = (a)dy_1$. And therefore, the ratio, $dy/dx = dy_1/dx_1$. So, the left-hand side becomes dy_1 over dx_1, and the right-hand side becomes F of, well, y over x is the same as y over, since I've scaled them equally, this is the same as y_1 over x_1.

So, it's y_1 over x_1, and the net effect is I simply, everywhere I have an x, I change it to x_1, and wherever I have a y, I change it to y_1, which, what's in a name? It's the identical equation. So, I haven't changed the equation at all via zoom transformation. And, that's what makes it homogeneous. That's not an uncommon use of the word homogeneous. When you say space is homogeneous, every direction, well, that means, I don't know. It means, okay, I'm getting into trouble there. I'll let it go since I can't prepare any better, I haven't prepared any better explanation, but this is a pretty good one.

Okay, so, suppose we've got a homogeneous equation. How do we solve it? So, here's our equation, F(y/x). Well, what substitution would you like to make? Obviously, we should make a direct substitution, z=y/x. So, why did he say that this was going to be an example of inverse substitution? Because I wanted to confuse you. But look, that's fine. If you write it in that form, you'll know exactly what to do with the right-hand side. And, this is why everybody loves to do that. But for Charlie, you have to substitute into the left-hand side as well.

And, I can testify, for many years of looking with sinking heart at examination papers, what happens if you try to make a direct substitution like this? You say, oh, I need z prime. z prime equals, well, I better use the quotient rule for differentiating that. And, it comes out this long, and then either a long pause, what do I do now? Because it's not at all obvious what to do at that point. Or, much worse, two pages of frantic calculations, and giving up in total despair.

Now, the reason for that is because you tried to do it making a direct substitution. All you have to do instead is use it, treat it as an inverse substitution, write y equals zx. What's the motivation for doing that? It's clear from the equation. This goes through all of mathematics. Whenever you have to change a variable, excuse me, whenever you have to change a variable, look at what you have to substitute for, and focus your attention on that.

I need to know what y prime is. Okay, well, then I better know what y is. If I know what y is, do I know what y prime is? Oh, of course. y' = z'x + z. And now, I turned with that one stroke, the equation has now become z'x + z + F(z). Well, I don't know. Can I solve that? Sure. That can be solved because this is x dz/dx. Just put the z on the other side, it's F(z) - z. And now, this side is just a function of z. Separate variables.

And, the only thing to watch out for is, at the end, the z was your business. You've got to put the answer back in terms of z and y. Okay, let's work an example of this. Since I haven't done any modeling yet this period, let's make a little model, differential equations model. It's a physical situation, which will be solved by an equation. And, guess what? The equation will turn out to be homogeneous.

Okay, so the situation is as follows. We are in the Caribbean somewhere, a little isolated island somewhere with a little lighthouse on it at the origin, and a beam of light shines from the lighthouse. The beam of light can rotate the way the lighthouse beams. But, this particular beam is being controlled by a guy in the lighthouse who can aim it wherever he wants. And, the reason he's interested in aiming it wherever he wants is there's a drug boat here, [LAUGHTER] which has just been caught in the beam of light.

So, the drug boat, which has just been caught in a beam of light, and feels it'd a better escape. Now, the lighthouse keeper wants to keep the drug boat; the light is shining on it so that the U.S. Coast Guard helicopters can zoom over it and do whatever they do to drug boats, -- -- I don't know. So, the drug boat immediately has to follow an escape strategy. And, the only one that occurs to him is, well, he wants to go further away, of course, from the lighthouse. On the other hand, it doesn't seem sensible to do it in a straight line because the beam will keep shining on him. So, he fixes the boat at some angle, let's say, and goes off so that the angle stays 45 degrees.

So, it goes so that the angle between the beam and maybe, draw the beam a little less like a 45 degree angle. So, the angle between the beam and the boat, the boat's path is always 45 degrees, goes at a constant 45 degree angle to the beam, hoping thereby to escape. On the other hand, of course, the lighthouse guy keeps the beam always on the boat. So, it's not clear it's a good strategy, but this is a differential equations class. The question is, what's the path of the boat? What's the boat's path? Now, an obvious question is, why is this a problem in differential equations at all? In other words, looking at this, you might scratch your head and try to think of different ways to solve it.

But, what suggests that it's going to be a problem in differential equations? The answer is, you're looking for a path. The answer is going to be a curve. A curve means a function. We are looking for an unknown function, in other words. And, what type of information do we have about the function? The only information we have about the function is something about its slope, that its slope makes a constant 45Â° angle with the lighthouse beam.

Its slope makes a constant known angle to a known angle. Well, if you are trying to find a function, and all you know is something about its slope, that is a problem in differential equations. Well, let's try to solve it. Well, let's see. Well, let me draw just a little bit. So, here's the horizontal. Let's introduce the coordinates. In other words, there's the horizontal and here's the boat to indicate where I am with respect to the picture.

So, here's the boat. Here's the beam, and the path of the boat is going to make a 45Â° angle with it. So, this is the path that we are talking about. And now, let's label what I know. Well, this angle is 45Â°. This angle, I don't know, but of course I can calculate it easily enough because it has to do with, if I know the coordinates of this point, (x, y), then of course that horizontal angle, I know the slope of this line, and that angle will be related to the slope.

So, let's call this alpha. And now, what I want to know is what the slope of the whole path is. So, y prime-- let's call y = y(x), the unknown function whose path, whose graph is going to be the boat's path, unknown graph. What's its slope? Well, its slope is the tangent of the sum of these two angles, alpha plus 45Â°. Now, what do I know? Well, I know that the tangent of alpha is how much? That's tan alpha = y/x. In other words, if this was the point, x over y, this is the angle it makes with a horizontal, if you think of it over here. So, this angle is the same as that one, and it's y over, its slope of that line is y over x.

So, the tangent of the angle is y over x. How about the tangent of 45Â°? That's one, and there's a formula. This is the hard part. All you have to know is that the formula exists, and then you will look it up if you have forgotten it, relating the tangent or giving you the tangent of the sum of two angles, and you can, if you like, clever, derive it from the formula for the sign and cosine of the sum of two angles. But, one peak is worth a thousand finesses.

So, it is the tangent of alpha plus the tangent of 45Â°. Let me read it out in all its gory details, divided by one, so you'll at least learn the formula, 1 - tan alpha * tan 45Â°. This would work for the tangent of the sum of any two angles. That's the formula. So, what do I get then? y' = (y/x + 1) / (1 - y/x).

Now, there is no reason for doing anything to it, but let's make it look a little prettier, and thereby, make it less obvious that it's a homogeneous equation. If I multiply top and bottom by x, it looks prettier. (x+y)/(x-y) = y'. That's our differential equation. But, notice, that let step to make it look pretty has undone the good work. It's fine if you immediately recognize this as being a homogeneous equation because you can divide the top and bottom by x. But here, it's a lot clearer that it's a homogeneous equation because it's already been written in the right form. Okay, let's solve it now, since we know what to do. We're going to use as the new variable, z = y/x.

And, as I wrote up there for y prime, we'll substitute z'x + z. And, with that, let's solve. Let's solve it. The equation becomes z'x + z = (z+1)/(1-z). We want to separate variables, so you have to put all the z's on one side. So, this is going to be x dz/dx = (z+1)/(1-z) - z. And now, as you realize, putting it on the other side, I'm going to have to turn it upside down. Just as before, if you have to turn something upside down, it's better to combine the terms, and make it one tiny little fraction.

Otherwise, you are in for quite a lot of mess if you don't do this nicely. So, z plus one minus z, that gets rid of the z's. (1 + z^2) / (1 - z). And so, the question is dz, and put this on the other side and turn it upside down. So, that will be (one minus z) over (one plus z squared) on the left-hand side and on the right-hand side, dx over x. Well, that's ready to be integrated just as it stands. The right-hand side integrates to be log x. The left-hand side is the sum of two terms. The integral of one over one plus z squared is the arc tangent of z, maybe?

The derivative of this is 1 / (1 + z^2). How about the term z / (1 + z^2)? Well, that integrates to be a logarithm. It is more or less the ln (1 + z^2). If I differentiate this, I get 1 / (1 + z^2) * 2z, but I wish I had negative z there instead. Therefore, I should put a minus sign, and I should multiply that by half to make it come out right.

And, this is log x on the right hand side plus, put in that arbitrary constant. And now what? Well, let's now fool around with it a little bit. The arc tangent, I'm going to simultaneously, no, two steps. I have to remember your innocence, although probably a lot of you are better calculators than I am. I'm going to change this, use as many laws of logarithms as possible. I'm going to put this in the exponent, and put this on the other side. That's going to turn it into ln(sqrt(1 + z^2)). And, this is going to be +ln(x) + c. And, now I'm going to make, go back and remember that z = y / x. So, this becomes the tan^(-1) (y/x) equals.

Now, I combine the logarithms. This is the log of x times this square root, right, make one logarithm out of it, and then put z = y/x. And, you see that if you do that, it'll be the ln(x sqrt(1 + (y/x)^2)), and what is that? Well, if I put this over x^2 and take it out, it cancels that. And, what you are left with is the ln(sqrt(x^2 + y^2)) + c. Now, technically, you have solved the equation, but not morally because, I mean, my God, what a mess! Incredible path. It tells me absolutely nothing. Wow, what is the screaming? Change me to polar coordinates. What's the arc tangent of y over x?

Theta. In polar coordinates it's theta. This is r. So, the curve is theta equals the log of r plus a constant. And, I can make even that little better if I exponentiate everything, exponentiate both sides, combine this in the usual way, the and what you get is that r = c1 e^(theta). That's the curve. It's called an exponential spiral, and that's what our little boat goes in. And notice, probably if I had set up the problem in polar coordinates from the beginning, nobody would have been able to solve it. But, anyone who did would have gotten that answer immediately. Thanks.