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18.03 Differential Equations, Spring 2006

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Today, once again, a day of solving no differential equations whatsoever. The topic is a special kind of differential equation, which occurs a lot. It's one in which the right-hand side doesn't have any independent variable in it. Now, since I'm going to use as the independent variable,  $t$  for time, maybe it would be better to write the left-hand side to let you know, since you won't be able to figure out any other way what it is,  $dy/dt$ .

We will write it this time.  $dy/dt$  is equal to, and the point is that there is no  $t$  on the right hand side. So, there's no  $t$ . There's a name for such an equation. Now, some people call it time independent. The only problem with that is that sometimes the independent variable is a time. It's something else. We need a generic word for there being no independent variable on the right-hand side. So, the word that's used for that is autonomous. So, that means no independent variable on the right-hand side.

It's a function of  $y$  alone, the dependent variable. Now, your first reaction should be, oh, well, big deal. Big deal. If there's no  $t$  on the right hand side, then we can solve this by separating variables. So, why has he been talking about it in the first place? So, I admit that. We can separate variables, and what I'm going to talk about today is how to get useful information out of the equation about how its solutions look without solving the equation. The reason for wanting to do that is, A, it's fast. It gives you a lot of insight, and the actual solution, I'll illustrate one, in the first place, take you quite a while.

You may not be able to actually do the integrations, the required and separation of variables to get an explicit solution, or it might simply not be worth the effort of doing if you only want certain kinds of separations about the solution. So, the thing is, the problem is, therefore, to get qualitative information about the solutions without actually solving -- Without actually having to solve the equation. Now, to do that, let's take a quick look at how the direction fields of such an equation, after all, it's the direction field is our principal tool for getting qualitative information about solutions without actually solving. So, how does the direction field look?

Well, think about it for just a second, and you will see that every horizontal line is an isocline. So, the horizontal lines, what are their equations? This is the  $t$  axis. And, here's the  $y$  axis. The horizontal lines have the formula  $y$  equals a constant. Let's make it  $y$  equals a  $y_0$  for different values of the constant  $y_0$ . Those are the horizontal lines. And, the point is they are isoclines. Why? Well, because along any one of these horizontal lines, I'll draw one in, what are the slopes of the line elements? The slopes are  $dy/dt$  is equal to  $f(y_0)$ , but that's a constant because there's no  $t$  to change as you move in the horizontal direction.

The slope is a constant. So, if I draw in that isocline, I guess I've forgotten, as our convention, isoclines are in dashed lines, but if you have color, you are allowed to put them in living yellow. Well, I guess I could make them solid, in that case. I don't have to make a dash. Then, all the line elements, you put them in at will because

they will all have, they are all the same, and they have slope, that,  $f(y_0)$ . And, similarly down here, they'll have some other slope. This one will have some other slope. Whatever, this is the  $y_0$ , the value of it, and whatever that happens to be. I'll put it one more. That's the x-axis.

can use the x-axis. That's an isocline, too. Now, what do you deduce about how the solutions must look? Well, let's draw one solution. Suppose one solution looks like this. Well, that's an integral curve, in other words. Its graph is a solution. Now, as I slide along, these slope elements stay exactly the same, I can slide this curb along horizontally, and it will still be an integral curve everywhere.

So, in other words, they integral curves are invariant under translation for an equation of this type. They all look exactly the same, and you get them all by taking one, and just pushing it along. Well, that's so simple it's almost uninteresting, except in that these equations occur a lot in practice. They are often hard to integrate directly. And, therefore, it's important to be able to get information about them. Now, how does one do that? There's one critical idea, and that is the notion of a critical point. These equations have what are called critical points. And, what it is is very simple. There are three ways of looking at it: critical point,  $y_0$ ; what does it mean for  $y_0$  to be a critical point?

It means, another way of saying it is that it should be a zero of the right-hand side. So, if I ask you to find the critical points for the equation, what you will do is solve the equation  $f(y) = 0$ . Now, what's interesting about them? Well, for a critical point, what would be the slope of the line element along, if this is at a critical level, if that's a critical point? Look at that isocline.

What's the slope of the line elements along it? It is zero. And therefore, for these guys, these are, in other words, our solution curves. But let's prove it formally. So, there are three ways of saying it.  $y_0$  is a critical point. It's a zero of the right-hand side, or,  $y$  equals  $y_0$  is a solution to the equation. Now, that's perfectly easy to verify. If  $y_0$  makes this right-hand side zero, it's certainly also  $y = y_0$  makes the left-hand side zero because you're differentiating a constant. So, the reasoning, if you want reasoning, is proof. Maybe we can make one line out of a proof.

To say that it's a solution, what does it mean to say that it's a solution? It means to say that when you plug it in, plug in this constant function,  $y_0$ , the  $dy_0 / dt = f(y_0)$ . Is that true? Yeah. Both sides are zero. It's true. Now,  $y_0$  is not a number. Well, it is. It's a number on this side, but on this side, what I mean is a constant function whose constant value is  $y_0$ , this function, and its derivatives are zero because it has slope zero everywhere.

So, this guy is a constant function, has slope zero. This is a number which makes the right-hand side zero. Well, that's nice. So, in other words, what we found are, by finding these critical points, solving that equation, we found all the horizontal solutions. But, what's so good about those? Surely, they must be the most interesting solutions there are. Well, think of how the picture goes. Let's draw in one of those horizontal solutions. So, here's a horizontal solution.

That's a solution. So, this is my  $y_0$ . That's the height at which it is. And, I'm assuming that  $f(y_0) = 0$ . So, that's a solution. Now, the significance of that is, because it's a solution, in other words, it's an integral curve, remember what's true about integral curves. Other curves are not allowed to cross them. And therefore,

these things are the absolute barriers. So, for example, suppose I have two of them is  $y_0$ , and let's say here's another one, another constant solution. I want to know what the curves in between those can do. Well, I do know that whatever those red curves do, the other integral curves, they cannot cross this, and they cannot cross that.

And, you must be able to translate them along each other without ever having two of them intersect. Now, that really limits their behavior, but I'm going to nail it down even more. So, other curves can't cross these. Other integral curves can't cross these yellow curves, these yellow lines, these horizontal lines. But, I'm going to show you more, and namely, so what I'm after is deciding, without solving the equation, what those curves must look like in between. Now, the way to do that is you draw, so if we want to make steps, everybody likes steps, okay, so step one is going to be, find these. Find the critical points.

And, you're going to do that by solving this equation, finding out where it's zero. Once you have done that, you are going to draw the graph of  $f(y)$ . And, the interest is going to be, where is it positive? Where is it negative? You've already found where it's zero. Everywhere else, therefore, it must be either positive or negative. Now, once you have found that out, why am I interested in that? Well, because  $dy/dt = f(y)$ , right? That's what the differential equation says. Therefore, if this, for example, is positive, that means this must be positive. It means that  $y$  must be increasing. It means the solution must be increasing.

Where it's negative, the solution will be decreasing. And, that tells me how it's behaving in between these yellow lines, or on top of them, or on the bottom. Now, at this point, I'm going to stop, or not stop, I mean, I'm going to stop talking generally. And everything in the rest of the period will be done by examples which will get increasingly complicated, not terribly complicated by the end.

But, let's do one that's super simple to begin with. Sorry, I shouldn't say that because some of you may be baffled even by here because after all I'm going to be doing the analysis not in the usual way, but by using new ideas. That's the way you make progress. All right, so, let's do our bank account. So,  $y$  is money in the bank account.  $r$  is the interest rate. Let's assume it's a continuous interest rate. All banks nowadays pay interest continuously, the continuous interest rate. So, if that's all there is, and money is growing, you know the differential equation says that the rate at which it grows is equal to  $r$ , the interest rate times a principle, the amount that's in the bank at that time.

So, that's the differential equation that governs that. Now, that's, of course, the solution is simply an exponential curve. There's nothing more to say about it. Now, let's make it more interesting. Let's suppose there is a shifty teller at the bank, and your money is being embezzled from your account at a constant rate. So, let's let  $w$  equal, or maybe  $e$ , but  $e$  has so many other uses in mathematics,  $w$  is relatively unused,  $w$  is the rate of embezzlement, thought of as continuous. So, every day a little bit of money is sneaked out of your account because you are not paying any attention to it. You're off skiing somewhere, and not noticing what's happening to your bank account.

So, since it's the rate, the time rate of embezzlement, I simply subtract it from this. It's not  $w$  times  $y$  because the embezzler isn't stealing a certain fraction of your account. It's simply stealing a certain number of dollars every day, the same number

of dollars being withdrawn for the count. Okay, now, of course, you could solve this. This separates variables immediately. You get the answer, and there's no problem with that. Let's analyze the behavior of the solutions without solving the equation by using these two points. So, I want to analyze this equation using the method of critical points. So, the first thing I should do is, so, here's our equation, is find the critical points.

Notice it's an autonomous equation all right, because there's no  $t$  on the right-hand side. Okay, so, the critical points, well, that's where  $ry$  minus  $w$  equals zero. In other words, there's only one critical point, and that occurs when  $y$  is equal to  $w$  over  $r$ . So, that's the only critical point. Now, I want to know what's happening to the solution. So, in other words, if I plot, I can write away, of course, negative values aren't of particularly interesting here, there is definitely a horizontal solution, and it has the value, it's at the height,  $y = w / r$ . That's a solution.

The question is, what do the other solutions look like? Now, watch how I make the analysis because I'm going to use two now. So, this is step one, then step two. What do I do? Well, I'm going to graph  $f(y)$ . Well,  $f(y) = ry - w$ . What does that look like? Okay, so, here is the  $y$ -axis. Notice the  $y$ -axis is going horizontally because what I'm interested in is the graph of this function.

What do I call the other axis? I'm going to use the same terminology that is used on the little visual that describes this. And, that's  $dy$ . You could call this other axis the  $f(y)$  axis. That's not a good name for it. You could call it the  $dy / dt$  axis because it's, so to speak, the other variable. That's not great either. But, worst of all would be introducing yet another letter for which we would have no use whatsoever.

So, let's think of it. We are plotting, now, the graph of  $f(y)$ .  $f(y) = ry - w$ . Well, that's a line. Its intercept is down here at  $w$ , and so the graph looks something like this. It's a line. This is the line,  $ry - w$ . It has slope  $r$ . Well, what am I going to get out of that line? Just exactly this. What am I interested in about that line? Nothing other than where is it above the axis, and where is it below? This function is positive over here, and therefore, I'm going to indicate that symbolically, this, by putting an arrow here.

The meaning of this arrow is that  $y(t)$  is increasing. See where it's the right-hand side of that last board?  $y$  of  $t$  is increasing when  $f(y)$  is positive.  $f$  of  $y$  is positive here, and therefore, to the right of this point, it's increasing. Here to the left of it,  $f$  of  $y$  is negative, and therefore over here it's going to be decreasing. What point is this, in fact? Well, that's where it crosses the axis. That's exactly the critical point,  $w / r$ . Therefore, what this says is that a solution, once it's bigger than  $y / r$ , it increases, and it increases faster and faster because this function is higher and higher.

And, that represents the rate of change. So, in other words, once the solution, let's say a solution starts over here at time zero. So, this is the  $t$  axis. And, here is the  $y$  axis. So, now, I'm plotting solutions. If it starts at  $t$  equals zero, above this line, that is, starts with the value  $w / r$ , which is bigger than zero, a value bigger than  $w$  over  $r$ , then it increases, and increases faster and faster.

If it starts below that, it decreases and decreases faster and faster. Now, in fact, I only have to draw two of those because what do all the others look like? They are translations. All the other curves look exactly like those. They are just translations of them. This guy, if I start closer, it's still going to decrease. Well, that's supposed to

be a translation. Maybe it is. So, these guys look like that. Let's do just a tiny bit more interpretation of that.

Well, I think I better leave it there because we've got harder things to do, and I want to make sure we've got time for it. Sorry. Okay, next example, a logistic equation. Some of you have already solved this in recitation, and some of you haven't. This is a population equation. This is the one that section 7.1 and section 1.7 is most heavily concerned with, this particular equation. The derivation of it is a little vague. It's an equation which describes how population increases. And one minute, the population behavior of some population, --

-- let's call it,  $y$  is the only thing I know to call anything today, but of course your book uses capital  $P$  for population, to get you used to different variables. Now, the basic population equation runs  $dy / dt$ . There's a certain growth rate. Let's call it  $k$ . So,  $k$  is what's called the growth rate. It's actually, sometimes it's talked about in terms of birthrate. But, it's the net birth rate. It's the rate at which people, or bacteria, or whatever are being born minus the rate at which they are dying. So, it's a net birthrate. But, let's just call it the growth rate. Now, if this is the equation, we can think of this, if  $k$  is constant, that's what's called simple population growth. And you are all familiar with that. Logistical growth allows for slightly more complex situations.

Logistic growth says that calling  $k$  a constant is unrealistic because the Earth is not filled entirely with people. What stops it from having unlimited growth? Well, the fact that the resources, the food, the organism has to live on gets depleted. And, in other words, the growth rate declines as  $y$  increases. As the population increases, one expects the growth rate to decline because resources are being used up, and they are not indefinitely available.

Well, in other words, we should replace  $k$  by a function with this behavior. What's the simplest function that declines as  $y$  increases? The simplest choice, and if you are ignorant about what else to do, stick with the simplest, at least you won't work any harder than you have to, would be to take  $k$  equal to the simplest declining function of  $y$  there is, which is simply a linear function,  $A - by$ .

So, if I use that as the choice of the declining growth rate, the new equation is  $dy / dt$  equals, here's my new  $k$ . The  $y$  stays the same, so the equation becomes  $dy / dt = (a - by)y$ , or in other words,  $ay - by^2$ . This equation is what's called the logistic equation. It has many applications, not just to population growth. It's applied to the spread of disease, the spread of a rumor, the spread of many things. Yeah, a couple pieces of chalk here.

Okay, now, those of you who have solved it know that the explicit solution involves, well, you separate variables, but you will have to use partial fractions, ugh, I hope you love partial fractions. You're going to need them later in the term. But, I could avoid them now by not solving the equation explicitly. But anyway, you get a solution, which I was going to write on the board for you, but you could look it up in your book. It's unpleasant enough looking to make you feel that there must be an easier way at least to get the basic information out. Okay, let's see if we can get the basic information out. What are the critical points? Well, this is pretty easy. A, I want to set the right-hand side equal to zero. So, I'm going to solve the equation.

I can factor out a  $y$ . It's going to be  $y(a-by) = 0$ . And therefore, the critical points are where  $y = 0$ . That's one. And, the other factor is when this factor is zero, and that happens when  $y = a/b$ . So, there are my two critical points. Okay, what does, let's start drawing pictures of solutions. Let's put it in those right away. Okay, the critical point, zero, gives me a solution that looks like this. And, the critical point,  $a/b$ , those are positive numbers. So, that's somewhere up here. So, those are two solutions, constant solutions. In other words, if the population by dumb luck started at zero, it would stay at zero for all time.

That's not terribly surprising. But, it's a little less obvious that if it starts at that magic number,  $a/b$ , it will also stay at that magic number for all time without moving up or down or away from it. Now, the question is, therefore, what happens in between? So, for the in between, I'm going to make that same analysis that I made before. And, it's really not very hard. Look, so here's my  $dy/dt$ -axis. I'll call that  $y'$ , okay? And, here's the  $y$ -axis. So, I'm now doing step two. This was step one. Okay, the function that I want to graph is this one,  $ay - by^2$ , or in factor form,  $y(a - by)$ . Now, this function, we know, has a zero. It has a zero here, and it has a zero at the point  $a/b$ .

At these two critical points, it has a zero. What is it doing in between? Well, in between, it's a parabola. It's a quadratic function. It's a parabola. Does it go up or does it go down? Well, when  $y$  is very large, it's very negative. That means it must be a downward-opening parabola. And therefore, this curve looks like this. So, I'm interested in knowing, where is it positive, and where is it negative? Well, it's positive, here, for this range of values of  $y$ . Since it's positive there, it will be increasing there. Here, it's negative, and therefore it will be decreasing.

Here, it's negative, and therefore,  $dy/dt$  will be negative also, and therefore the function,  $y$ , will be decreasing here. So, how do these other solutions look? Well, we can put them in. I'll put them in in white, okay, because this has got to last until the end of the term. So, how are they doing? They are increasing between the two curves. They are not allowed to cross either of these yellow curves. But, they are always increasing. Well, if they're always increasing, they must start here and increase, and not allowed to cross. It must do something like that. This must be a translation of it. In other words, the curves must look like that. Those are supposed to be translations of each other.

I know they aren't, but use your imaginations. But what's happening above? So in other words, if I start with a population anywhere bigger than zero but less than  $a/b$ , it increases asymptotically to the level  $a/b$ . What happens if I start above that? Well, then it decreases to it because, this way, for the values of  $y > a/b$ , it decreases as time increases. So, these guys up here are doing this.

And, how about the ones below the axis? Well, they have no physical significance. But let's put them in anyway. Whether they doing? They are decreasing away from zero. So, these guys don't mean anything physically, but mathematically they exist. Their solutions, they're going down like that. Now, you notice from this picture that there are, even though both of these are constant solutions, they have dramatically different behavior.

This one, this solution, is the one that all other solutions try to approach as time goes to infinity. This one, the solution zero, is repulsive, as it were. Any solution that starts near zero, if it starts at zero, of course, it stays there for all time, but if it

starts just a little bit above zero, it increases to  $a/b$ . This is called a stable solution because everybody tries to get closer and closer to it. This is called, zero is also a constant solution, but this is an unstable solution. And now, usually, solution is too general a word. I think it's better to call it a stable critical point, and an unstable critical point. But, of course, it also corresponds to a solution.

So, critical points are not all the same. Some are stable, and some are unstable. And, you can see which is which just by looking at this picture. If the arrows point towards them, you've got a stable critical point. If it arrows point away from them, you've got an unstable critical point. Now, there is a third possibility. Okay, I think we'd better address it because otherwise you're going to sit there wondering, hey, what did he do? Suppose it looks like this. Suppose it were just tangent. Well, this is the picture of that curve, the pink curve. What would the arrows look like then?

What would the arrows look like then? Well, since they are positive, it's always positive, the arrow goes like this. And then on the side, it also goes in the same direction. So, is this critical point stable or unstable? It's stable if you approach it from the left. So, how, in fact, do the curves, how would the corresponding curves look? Well, there's our long-term solution. This corresponds to that point. Let's call this  $a$ , and then this will be the value,  $a$ . If I start below it, I rise to it. If I start above it, I increase. So, if I start above it, I do this. Well, now, that's stable on one side, and unstable on the other. And, that's indicated by saying it's semi-stable.

That's a brilliant word. I wonder how long it to do think that one up, semi-stable critical point: stable on one side, unstable on the other depending on whether you start below it. And, of course, it could be reversed if I had drawn the picture the other way. I could have approached it from the top, and left it from below. You get the idea of the behavior. Okay, let's now take, I'm going to soup up this logistic equation just a little bit more. So, let's talk about the logistic equation. But, I'm going to add to it harvesting, with harvesting.

So, this is a very late 20th century concept. So, we imagine, for example, a bunch of formerly free range Atlantic salmon penned in one of these huge factory farms off the coast of Maine or someplace. They've made salmon much cheaper than it used to be, but at a certain cost to the salmon, and possibly to our environment. So, what happens? Well, the salmon grow, and grow, and do what salmon do. And, they are harvested. That's a word somewhere in the category of ethnic cleansing in my opinion. But, it's, again, a very 20th-century word. I think it was Hitler who discovered that, that all you had to do was call something by a sanitary name, and no matter how horrible it was, good bourgeois people would accept it.

So, the harvesting, which means, of course, picking them up and killing them, and putting them in cans and stuff like that, okay, so what's the equation? I'm going to assume that the harvest is at a constant time rate. In other words, it's not a certain fraction of all the salmon that are being caught each day and canned. The factory has a certain capacity, so, 400 pounds of salmon each day are pulled out and canned. So, it's a constant time rate. That means that the equation is now going to be  $dy/dt$  is equal to, well, salmon grow logistically.

$ay - by^2$ , so, that part of the equation is the same. But, I need a term to take care of this constant harvesting rate, and that will be  $h$ . Let's call it  $h$ , not  $h$  times  $y$ . Then, I would be harvesting a certain fraction of all the salmon there, which is not what I'm doing. Okay: our equation. Now, I want to analyze what the critical points



of this look like. Now, this is a little more subtle because there's now a new parameter, there. And, what I want to see is how that varies with the new parameter. The best thing to do is, I mean, the thing not to do is make this equal to zero, fiddle around with the quadratic formula, get some massive expression, and then spend the next half hour scratching your head trying to figure out what it means, and what information you are supposed to be getting out of it.

Draw pictures instead. Draw pictures. If  $h$  is zero, that's the smallest harvesting rate I could have. The picture looks like our old one. So, if  $h$  is zero, the picture looks like, what color did I, okay, pink. Yellow. Yellow is the cheapest, but I can't find it. Okay, yellow is commercially available. These are precious. All right, purple if it's okay, purple. So, this is the one, our original one corresponding to  $h$  equals zero. Or, in other words, it's the equation  $ay - by^2$ .  $h$  is zero. Now, if I want to find, I now want to increase the value of  $h$ , well, if I increase the value of  $h$ , in other words, harvest more and more, what's happening?

Well, I simply lower this function by  $h$ . So, if I lower  $h$  somewhat, it will come to here. So, this is some value,  $ay - by^2 - h$ , let's say. That's this curve. If I lower it a lot, it will look like this. So,  $ay - by^2 - h > 0$ . This doesn't mean anything. Two. Obviously, there's one interesting value to lower it by. It's a value which would lower it exactly by this amount. Let me put that in special. If I lower it by just that amount, the curve always looks the same.

It's just been lowered. I'm going to say this one is, so this one is the same thing, except that I've subtracted  $(h)m$ . Where is  $(h)m$  on the picture? Well, I lowered it by this amount. So, this height is  $h$  sub  $m$ . In other words, if I find the maximum height here, which is easy to do because it's a parabola, and lower it by exactly that amount, I will have lowered it to this point.

This will be a critical point. Now, the question is, what's happened to the critical point as I did this? I started with the critical points here and here. As I lower  $h$ , the critical point changed to this and that. And now, it changed to this one when I got to the purple line. And, as I went still further down, there were no critical points. So, this curve has no critical points attached to it. What are the corresponding pictures? Well, the corresponding pictures, well, we've already drawn, the picture for  $h = 0$  is drawn already. The pictures that I'm talking about are how the solutions look. How would the solution look like for this one for  $h_1$ ? For  $h_1$ , the solutions look like, here is  $a/b$ .

Here is  $a/b$ , but the critical points aren't at zero and  $a$  over  $b$  anymore. They've moved in a little bit. So, they are here and here. And, otherwise, the solutions look just like they did before, and the analysis is the same. And, similarly, if  $h_2$  goes very far, if  $h_2$  is very large, there are no critical points.  $h$ , too large, no critical points. Are the solutions decreasing all the time or increasing? Well, they are always decreasing because the function is always negative. Solutions always go down, always. The interesting one is this last one, where I decreased it just to  $(h)m$ . And, what happens there is there is this certain, magic critical point whose value we could calculate.

here's one constant solution. So, this is one that has the value. Sorry, I'm calculating the solutions out. So,  $y$  here and  $t$  here, so here it is value,  $(h)m$  is the value by which it has been lowered. So, this is the picture for  $(h)m$ . And, how do the solutions look? Well, to the right of that, they are decreasing. And, to the left they are also decreasing because this function is always negative. So, the solutions look like this, if you start above, and if you start below, they decrease. And, of course, they can't get

lower than zero because these are salmon. What is the significance of  $(h)_m$ ?  $(h)_m$  is the maximum rate of harvesting. It's an extremely important number for this industry.

If the maximum time rate at which you can pull the salmon daily out of the water, and can them without what happening? Without the salmon going to zero. As long as you start above, and don't harvest it more than this rate, it will be following these curves. You will be following these curves, and you will still have salmon. If you harvest just a little bit more, you will be on this curve that has no critical points, and the salmon in the tank will decrease to zero.