y' and y''. So, q(x)*y equal zero. The linearity of the equation, that is, the form in which it appears is going to be the key idea today. Today is going to be theoretical, but some of the ideas in it are the most important in the course. So, I don't have to apologize for the theory. Remember, the solution method was to find two independent y1, y2 independent solutions. And now, I'll formally write out what independent means.

There are different ways to say it. But for you, I think the simplest and most intelligible will be to say that y2 is not to be a constant multiple of y1. And, unfortunately, it's necessary to add, nor is y1 to be a constant multiple. I have to call it by different constants. So, let's call this one c prime of y2. Well, I mean, the most obvious question is, well, look. If this is not a constant times that, this can't be there because I would just use one over c if it was. Unfortunately, the reason I have to write it this way is to take account of the possibility that y1 might be zero. If y1 is zero, so, the bad case that must be excluded is that y1 equals zero, y2 nonzero.

I don't want to call those independent. But nonetheless, it is true that y2 is not a constant multiple of y1. However, y1 is a constant multiple of y2, namely, the multiple zero. It's just to exclude that case that I have to say both of those things. And, one would not be sufficed. That's a fine point that I'm not going to fuss over. But I just have, of course. Now, why do you do that? That's because, then, all solutions, and this is what concerns us today, are what? The linear combination with constant coefficients of these two, and the fundamental question we have to answer today is, why?

Now, there are really two statements involved in that. On the one hand, I'm claiming there is an easier statement, which is that they are all solutions. So, that's question one, or statement one. Why are all of these guys solutions? That, I could trust you to answer yourself. I could not trust you to answer it elegantly. And, it's the elegance that's the most important thing today because you have to answer it elegantly. Otherwise, you can't go on and do more complicated things. If you answer it in an ad hoc basis just by hacking out a computation, you don't really see what's going on. And you can't do more difficult things later.

So, we have to answer this, and answer it nicely. The second question is, so, if that answers why there are solutions at all, why are they all the solutions? Why all the solutions? In other words, to say it as badly as possible, why are all solutions, why all the solutions-- Never mind. Why are all the solutions. This is a harder question to answer, but that should make you happy because that means it depends upon a theorem which I'm not going to prove.

I'll just quote to you. Let's attack there for problem one first. q1 is answered by what's called the superposition. The superposition principle says exactly that. It says exactly that, that if y1 and y2 are solutions to a linear equation, to a linear homogeneous ODE, in fact it can be of higher order, too, although I won't stress that. In other words, you don't have to stop with the second derivative. You could add a third derivative and a fourth derivative. As long as the former makes the same, but that implies automatically that c_1 y_1 plus c_2 y_2 is a solution.

Now, the way to do that nicely is to take a little detour and talk a little bit about linear operators. And, since we are going to be using these for the rest of the term, this is the natural place for you to learn a little bit about what they are. So, I'm going to do it. Ultimately, I am aimed at a proof of this statement, but there are going to be certain side excursions I have to make. The first side side excursion is to write the differential equation in a different way. So, I'm going to just write its transformations. The first, I'll simply recopy what you know it to be, q y equals zero. That's the first form. The second form, I'm going to replace this by the differentiation operator.

So, I'm going to write this as $D^2(y)$. That means differentiate it twice. D it, and then D it again. This one I only have to differentiate once, so I'll write that as p $D(y)$, p times the derivative of Y. The last one isn't differentiated at all. I just recopy it. Now, I'm going to formally factor out the y. So this, I'm going to turn into $D^2 + pD + q$.

Now, everybody reads this as times y equals zero. But, what it means is this guy, it means this is shorthand for that. I'm not multiplying. I'm multiplying q times y. But, I'm not multiplying D times y. I'm applying D to y. Nonetheless, the notation suggests this is very suggestive of that. And this, in turn, implies that. I'm just transforming it. And now, I'll take the final step. I'm going to view this thing in parentheses as a guy all by itself, a linear operator. This is a linear operator, called a linear operator. And, I'm going to simply abbreviate it by the letter L. And so, the final version of this equation has been reduced to nothing but $Ly = 0$.

Now, what's L? You can think of L as, well, formally, L you would write as $D^2 + pD + q$. But, you can think of L, the way to think of it is as a black box, a function of what goes into the black box, well, if this were a function box, what would go it would be a number, and what would come out with the number. But it's not that kind of a black box. It's an operator box, and therefore, what goes in is a function of x. And, what comes out is another function of x, the result of applying this operator to that. So, from this point of view, differential equations, trying to solve the differential equation means, what should come out you want to come out zero, and the question is, what should you put in?

That's what it means solving differential equations in an inverse problem. The easy thing is to put it a function, and see what comes out. You just calculate. The hard thing is to ask, you say, I want such and such a thing to come out, for example, zero; what should I put in? That's a difficult question, and that's what we're spending the term answering. Now, the key thing about this is that this is a linear operator.

And, what that means is that it behaves in a certain way with respect to functions. The easiest way to say it is, I like to make two laws of it, that L of u_1, if you have two functions, I'm not going to put up the parentheses, x, because that just makes things look longer and not any clearer, actually. What does L do to the sum of two functions? If that's a linear operator, if you put in the sum of two functions, what you must get out is the corresponding L's, the sum of the corresponding L's of each. So, that's a law. And, the other law, linearity law, and this goes for anything in mathematics and its applications, which is called linear, basically anything is linear if it does the following thing to functions or numbers or whatever.

The other one is of a constant times any function, I don't have to give it a number now because I'm only using one of them, should be equal to c times L of u. So, here, c is a constant, and here, of course, the u is a function, functions of x. These are the two laws of linearity. An operator is linear if it satisfies these two laws. Now, for example, the differentiation operator is such an operator. D is linear, why? Well, because of the very first things you verify after you learn what the derivative is because the derivative of, well, I will write it in the D form. I'll write it in the form in which you know it. It would be $D(u_1 + u_2)$. How does one write that in ordinary calculus? Well, like that.

Or, maybe you write d by dx out front. Let's write it this way, is equal to $u_1' + u_2'$. That's a law. You prove it when you first study what a derivative is. It's a property. From our point of view, it's a property of the differentiation operator. It has this property. The D of u1 plus u2 is $D(u_1 + u_2) = D(u_1) + D(u_2)$. And, it also has the property that $c u'$, you can pull out the constant.

That's not affected by the differentiation. So, these two familiar laws from the beginning of calculus say, in our language, that D is a linear operator. What about the multiplication of law? That's even more important, that u1 times u2 prime, I have nothing whatever to say about that in here. In this context, it's an important law, but it's not important with respect to the study of linearity.

So, there's an example. Here's a more complicated one that I'm claiming is the linear operator. And, since I don't want to have to work in this lecture, the work is left to you. So, the proof, prove that L is linear, is this particular operator. L is linear. That's in your part one homework to verify that. And you will do some simple exercises in recitation tomorrow to sort of warm you up for that if you haven't done it already.

Well, you shouldn't have because this only goes with this lecture, actually. It's forbidden to work ahead in this class. All right, where are we? Well, all that was a prelude to proving this simple theorem, superposition principle. So, finally, what's the proof? The proof of the superposition principle: if you believe that the operator is linear, then L of c1, in other words, the ODE is L. $L = D^2 + pD + q$.

So, the ODE is $Ly = 0$. And, what am I being asked to prove? I'm being asked to prove that if y1 and y2 are solutions, then so is that thing. By the way, that's called a linear combination. Put that in your notes. Maybe I better write it even on the board because it's something people say all the time without realizing they haven't defined it. This is called a linear combination.

This expression is called a linear combination of y1 and y2. It means that particular sum with constant coefficients. Okay, so, the ODE is Ly equals zero. And, I'm trying to prove that fact about it, that if y1 and y2 are solutions, so is a linear combination of them. So, the proof, then, I start with apply $L(c_1y_1 + c_2y_2)$. Now, because this operator is linear, it takes the sum of two functions into the corresponding sum up what the operator would be. So, it would be $L(c_1y_1) + L(c_2y_2)$. That's because L is a linear operator. But, I don't have to stop there. Because L is a linear operator, I can pull the c out front. So, it's $c_1 L(y_1) + c_2 L(y_2)$.

Now, where am I? Trying to prove that this is zero. Well, what is L of y1? At this point, I use the fact that y1 is a solution. Because it's a solution, this is zero. That's what it means to solve that differential equation. It means, when you apply the linear operator, L, to the function, you get zero. In the same way, y2 is a solution. So, that's zero. And, the sum of c1*0 + c2*0 = 0. That's the argument. Now, you could make the same argument just by plugging c1 y1, plugging it into the equation and calculating, and calculating, and calculating, grouping the terms and so on and so forth. But, that's just calculation. It doesn't show you why it's so. Why it's so is because the operator, this differential equation is expressed as a linear operator applied to y is zero.

And, the only properties that are really been used as the fact that this operator is linear. That's the key point. L is linear. It's a linear operator. Well, that's all there is to the superposition principle. As a prelude to answering the more difficult question, why are these all the solutions? Why are there no other solutions? We need a few definitions, and a few more ideas. And, they are going to occur in connection with, so I'm now addressing, ultimately, question two. But, it's not going to be addressed directly for quite awhile. Instead, I'm going to phrase it in terms of solving the initial value problem.

So far, we've only talked about the general solution with those two arbitrary constants. But, how do you solve the initial value problem, in other words, fit initial conditions, find the solution with given initial values for the function and its derivatives. Now, -- -- the theorem is that this collection of functions with these arbitrary constants, these are all the solutions we have so far. In fact, they are all the solutions there are, but we don't know that yet. However, if we just focus on the big class of solutions, there might be others lurking out there somewhere lurking out there.

I don't know. But let's use what we have, that just from this family is enough to satisfy any initial condition, to satisfy any initial values. In other words, if you give me any initial values, I will be able to find the c1 and c2 which work. Now, why is that? Well, I'd have to do a song and dance at this point, if you hadn't been softened up by actually calculating for specific differential equations. You've had exercises and actually how to calculate the values of c1 and c2. So, I'm going to do it now in general what you have done so far for particular equations using particular values of the initial conditions.

So, I'm relying on that experience that you've had in doing the homework to make intelligible what I'm going to do now in the abstract using just letters. So, why is this so? Why is that so? Well, we are going to need it, by the way, here, too. I'll have to, again, open up parentheses. But let's go as far as we can. Well, you just try to do it. Suppose the initial conditions are, how will we write them?

So, they're going to be at some initial point, x0. You can take it to be zero if you want, but I'd like to be, just for a little while, a little more general. So, let's say the initial conditions, the initial values are being given at the point x0, all right, that's going to be some number. Let's just call it a. And, the initial value also has to specify the velocity or the value of the derivative there.

Let's say these are the initial values. So, the problem is to find the c which work. Now, how do you do that? Well, you know from calculation. You write y equals c1 y1 plus c2 y2. And you write y prime, and you take the derivative underneath that, which is easy to do. And now, you plug in x equals zero. And, what happens? Well, these now turn into a set of equations. What will they look like? Well, y of x0 is a, and this is b. So, what I get is let me flop it over onto the other side because that's the way you're used to looking at systems of equations. So, what we get is c1 y1(x0) + c2 y2(x0).

What's that supposed to be equal to? Well, that's supposed to be equal to y(x0). It's supposed to be equal to the given number, a. And, in the same way, c1 y1'(x0) + c2 y2'(x0) = b. In the calculations you've done up to this point, y1 and y2 were always specific functions like e^x or cos(22x), stuff like that.

Now I'm doing it in the abstract, just calling them y1 and y2, so as to include all those possible cases. Now, what am I supposed to do? I'm supposed to find c1 and c2. What kind of things are they? This is what you studied in high school, right? The letters are around us, but it's a pair of simultaneous linear equations. What are the variables? What are the variables? What are the variables? Somebody raise their hand.

If you have a pair of simultaneous linear equations, you've got variables and you've got constants, right? And you are trying to find the answer. What are the variables? Yeah? c1 and c2. Very good. I mean, it's extremely confusing because in the first place, how can they be the variables if they occur on the wrong side? They're on the wrong side; they are constants. How can constants be variables? Everything about this is wrong. Nonetheless, the c1 and the c2 are the unknowns, if you like the high school terminology. c1 and c2 are the unknowns. These messes are just numbers. After you've plugged in x0, this is some number. You've got four numbers here. So, c1 and c2 are the variables. The two find, in other words, to find the values of.

All right, now you know general theorems from 18.02 about when can you solve such a system of equations. I'm claiming that you can always find c1 and c2 that work. But, you know that's not always the case that a pair of simultaneous linear equations can be solved. There's a condition. There's a condition which guarantees their solution, which is what? What has to be true about the coefficients? These are the coefficients. What has to be true? The matrix of coefficients must be invertible. The determinant of coefficients must be nonzero. So, they are solvable if, for the c1 and c2, if this thing, I'm going to write it. Since all of these are evaluated at x0, I'm going to write it in this way.

y1, the determinant, whose entries are |y1, y2; y1', y2'|, evaluated at zero, x0, that means that I evaluate each of the functions in the determinant at x0. I'll write it this way. That should be not zero. So, in other words, the key thing which makes this possible, makes it possible for us to solve the initial value problem, is that this funny determinant should not be zero at the point at which we are interested. Now, this determinant is important in 18.03. It has a name, and this is when you're going to learn it, if you don't know it already. That determinant is called the Wronskian.

The Wronskian of what? If you want to be pompous, you say this with a V sound instead of a W. But, nobody does except people trying to be pompous. The Wronskian, we'll write a W. Now, notice, you can only calculate it when you know what the two functions are. So, the Wronskian of the two functions, y1 and y2, what's the variable? It's not a function of two variables, y1 and y2. These are just the names of functions of x.

So, when you do it, put it in, calculate out that determinant. This is a function of x, a function of the independent variable after you've done the calculation. Anyway, let's write its definition, |y1, y2; y1', y2'|. Now, in order to do this, the point is we must know that that Wronskian is not zero, that the Wronskian of these two functions is not zero at the point x0.

Now, enter a theorem which you're going to prove for homework, but this is harder. So, it's part two homework. It's not part one homework. In other words, I didn't give you the answer. You've got to find it yourself, alone or in the company of good friends. So, anyway, here's the Wronskian. Now, what can we say for sure? Note, suppose y1 and y, just to get you a feeling for it a little bit, suppose they were not independent. The word for not independent is dependent. Suppose they were dependent. In other words, suppose that y2 were a constant multiple of y1.

We know that's not the case because our functions are supposed to be independent. But suppose they weren't. What would the value of the Wronskian be? If y2 is a constant times y1, then y2 prime is that same constant times y1 prime. What's the value of the determinant? Zero. For what values of x is it zero for all values of x? And now, that's the theorem that you're going to prove, that if y1 and y2 are solutions to the ODE, I won't keep, say, it's the ODE we've been talking about, y'' + py.

But the linear homogeneous with not constant coefficients, just linear homogeneous second order. Our solutions, as there are only two possibilities, either-or. Either the Wronskian of y, there are only two possibilities. Either the Wronskian of y1 and y2 is always zero, identically zero, zero for all values of x. This is redundant. When I say identically, I mean for all values of x. But, I am just making assurance doubly sure.

Or, or the Wronskian is never zero. Now, there is no notation as for that. I'd better just write it out, is never zero, i.e. for no x is it, i.e. for all x. There's no way to say that. I mean, for all values of x, it's not zero. That means, there is not a single point for which it's zero. In particular, it's not zero here. So, this is your homework: problem five, part two. I'll give you a method of proving it, which was discovered by the famous Norwegian mathematician, Abel, who is, I guess, the centenary of his birth, I guess, was just celebrated last year.

He has one of the truly tragic stories in mathematics, which I think you can read. It must be a Simmons book, if you have that. Simmons is very good on biographies. Look up Abel. He'll have a biography of Abel, and you can weep if you're feeling sad. He died at the age of 26 of tuberculosis, having done a number of sensational things, none of which was recognized in his lifetime because people buried his papers under big piles of papers. So, he died unknown, uncelebrated, and now he's Norway's greatest culture hero. In the middle of a park in Oslo, there's a huge statue.

And, since nobody knows what Abel looked like, the statue is way up high so you can't see very well. But, the inscription on the bottom says Niels Henrik Abel, 1801-1826 or something like that. Now, -- -- the choice, I'm still, believe it or not, aiming at question two, but I have another big parentheses to open. And, when I closed it, the answer to question two will be simple. But, I think it's very desirable that you get this second big parentheses. It will help you to understand something important.

It will help you on your problem set tomorrow night. I don't have to apologize. I'm just going to do it. So, the question is, the thing you have to understand is that when I write this combination, I'm claiming that these are all the solutions. I haven't proved that yet. But, they are going to be all the solutions The point is, there's nothing sacrosanct about the y1 and y2. This is exactly the same collection as a collection which I would write using other constants. Let's call them u1 and u2. They are exactly the same, where u1 and u2 are any other pair of linearly independent solutions.

Any other pair of independent solutions, they must be independent, either a constant multiple of each other. In other words, u1 is some combination, now I'm really stuck because I don't know how to, c1 bar, let's say, that means a special value of c1, and a special value of c2, and u2 is some other special value, oh my God, c1 double bar, how's that? The notation is getting worse and worse. I apologize for it. In other words, I could pick y1 and y2 and make up all of these. And, I'd get a bunch of solutions. But, I could also pick some other family, some other two guys in this family, and just as well express the solutions in terms of u1 and u2. Now, well, why is he telling us that?

Well, the point is that the y1 and the y2 are typically the ones you get easily from solving the equations, like e^x and e^(2x). That's what you've gotten, or cos(x) and sin(x), something like that. But, for certain ways, they might not be the best way of writing the solutions. There is another way of writing those that you should learn, and that's called finding normalized, the normalized. They are okay, but they are not normalized.

For some things, the normalized solutions are the best. I'll explain to you what they are, and I'll explain to you what they're good for. You'll see immediately what they're good for. Normalized solutions, now, you have to specify the point at which you're normalizing. In general, it would be x0, but let's, at this point, since I don't have an infinity of time, to simplify things, let's say zero.

It could be x0, any point would do just as well. But, zero is the most common choice. What are the normalized solutions? Well, first of all, I have to give them names. I want to still call them y. So, I'll call them capital Y1 and Y2. And, what they are, are the solutions which satisfy certain, special, very special, initial conditions. And, what are those? So, they're the ones which satisfy, the initial conditions for Y1 are, of course there are going to be guys that look like this. The only thing that's going to make them distinctive is the initial conditions they satisfy. Y1 has to satisfy at zero. Its value should be one, and the value of its derivative should be zero.

For Y2, it's just the opposite. Here, the value of the function should be zero at zero. But, the value of its derivative, now, I want to be one. Let me give you a trivial example of this, and then one, which is a little less trivial, so you'll have some feeling for what I'm asking for. Suppose the equation, for example, is y double prime plus, well, let's really make it simple. Okay, you know the standard solutions are y1 = cos(x), and y2 = sin(x). These are functions, which, when you take the second derivative, they turn into their negative. You know, you could go the complex roots are i and minus i, and blah, blah, blah, blah, blah, blah. If you do it that way, fine. But at some point in the course you have to be able to write down and, right away, oh, yeah, cos(x), sin(x).

Okay, what are the normalized things? Well, what's the value of this at zero? It is one. What's the value of its derivative at zero? Zero. This is Y1. This is the only case in which you locked on immediately to the normalized solutions. In the same way, this guy is Y2 because its value at zero is zero. It's value of its derivative at zero is one. So, this is Y2. Okay, now let's look at a case where you don't immediately lock on to the normalized solutions. Very simple: all I have to do is change the sign. Here, you know, think through $r^2 - 1 = 0$. The characteristic roots are plus one and minus one, right?

And therefore, the solution is e^x, and e^{-x}. So, the solutions you find by the usual way of solving it is $y_1 = e^x$, and $y_2 = e^{-x}$. Those are the standard solution. So, the general solution is of the form. So, the general solution is of the form $c_1 e^x + c_2 e^{-x}$. Now, what I want to find out is what is Y1 and Y2?

How do I find out what Y1 is? Well, I have to satisfy initial conditions. So, if this is y, let's write down here, if you can still see that, $y' = c_1 e^x - c_2 e^{-x}$. So, if I plug in, I want y of zero to be one, I want this guy at the point zero to be one. What equation does that give me? That gives me $c_1 + c_2$, c1 plus c2, plugging in x equals zero, equals the value of this thing at zero. So, that's supposed to be one. How about the other guy? The value of its derivative is supposed to come out to be zero. And, what is its derivative? Well, plug into this expression. It's $c_1 - c_2$.

Okay, what's the solution to those pair of equations? c_2 has to be equal to c_1. The sum of the two of them has to be one. Each one, therefore, is equal to one half. And so, what's the value of Y1? Y1, therefore, is the function where c1 and c2 are one half. It's the function $Y_1 = (e^x + e^{-x})/2$. In the same way, I won't repeat the calculation. You can do yourself. Same calculation shows that Y2, so, put in the initial conditions. The answer will be that $Y_2 = (e^x - e^{-x})/2$. These are the special functions. For this equation, these are the normalized solutions. They are better than the original solutions because their initial values are nicer.

Just check it. The initial value, when x is equal to zero, the initial value, this has is zero. Here, when x is equal to zero, the value of the function is zero. But, the value of its derivative, these cancel, is one. So, these are the good guys. Okay, there's no colored chalk this period. Okay, there was colored chalk. There's one. So, for this equation, these are the good guys. These are our best solutions. e^x and e^{-x} are good solutions. But, these are our better solutions. And, this one, of course, is the function which is called hyperbolic sine of x, and this is the one which is called hyperbolic cosine of x. This is one of the most important ways in which they enter into mathematics. And, this is why the engineers want them. Now, why do the engineers want normalized solutions?

Well, I didn't explain that. So, what's so good about normalized solutions? Very simple: if Y1 and Y2 are normalized at zero, let's say, then the solution to the IVP, in other words, the ODE plus the initial values, $y(0) = a$ and $y'(0) = b$. So, the ODE I'm not repeating. It's the one we've been talking about all term since the beginning of the period. It's the one with the $p(x)$ and $q(x)$. And, here are the initial values. I'm going to call them a and b. You can also call them, if you like, maybe that's better to call them y_0, as they are individual in the homework.

They are called, I'm using the, let's use those. What is the solution? I say the solution is, if you use y1 and y2, the solution is $y_0 Y_1 + y_0' Y_2$. In other words, you can write down instantly the solution to the initial value problem, if instead of using the functions, you started out with the little Y1 and Y2, you use these better functions.

The thing that's better about them is that they instantly solve for you the initial value problem. All you do is use this number, initial condition as the coefficient of Y1, and use this number as the coefficient of Y2. Now, just check that by looking at it. Why is that so? Well, for example, let's check. What is its value of this function at zero? Well, the value of this guy at zero is one. So, the answer is y0 times one, and the value of this guy at zero is zero. So, this term disappears. And, it's exactly the same with the derivative. What's the value of the derivative at zero? The value of the derivative of this thing is zero. So, this term disappears. The value of this derivative at zero is one.

And so, the answer is y0'. So, check, check, this works. So, these better solutions have the property, what's good about them, and why scientists and engineers like them, is that they enable you immediately to write down the answer to the initial value problem without having to go through this business, which I buried down here, of solving simultaneous linear equations. Okay, now, believe it or not, that's all the work. We are ready to answer question number two: why are these all the solutions? Of course, I have to invoke a big theorem. A big theorem: where shall I invoke a big theorem? Let's see if we can do it here.

The big theorem says, it's called the existence and uniqueness theorem. It's the last thing that's proved at the end of an analysis course, at which real analysis courses, over which students sweat for one whole semester, and their reward at the end is, if they are very lucky, and if they have been very good students, they get to see the proof of the existence and uniqueness theorem for differential equations. But, I can at least say what it is for the linear equation because it's so simple.

It says, so, the equation we are talking about is the usual one, homogeneous equation, and I'm going to assume, you have to have assumptions that p and q are continuous for all x. So, they're good-looking functions. Coefficients aren't allowed to blow up anywhere. They've got to look nice. Then, the theorem says there is one and only one solution, one and only solution satisfying, given initial values such that y of zero, let's say $y(0) = A$, and y, let's make y0, and $y'(0) = B$.

The initial value problem has one and only one solution. The existence is, it has a solution. The uniqueness is, it only has one solution. If you specify the initial conditions, there's only one function which satisfies them and at the same time satisfies that differential equation. Now, this answers our question. This answers our question, because, look, what I want is all solutions. What we want are all solutions to the ODE. And now, here's what I say: a claim that this collection of functions, c1Y1 + c2Y2 are all solutions.

Of course, I began a period by saying I'd show you that c1 little y1 c little y2 are all the solutions. But, it's the case that these two families are the same. So, the family that I started with would be exactly the same as the family c1' Y1 because, after all, these are two special guys from that collection. So, it doesn't matter whether I talk about the original ones, or these. The theorem is still the same. The final step, therefore, if you give me one more minute, I think that will be quite enough. Why are these all the solutions?

Well, I have to take an arbitrary solution and show you that it's one of these. So, the proof is, given a solution, u(x), what are its values? Well, u(x0) = u0, and u'(x0), zero, let's say, is equal to u(0), is equal to some other number. Now, what's the solution? Write down what's the solution of these using the Y1's? Then, I know I've just shown you that u0Y1 + u0' Y2 satisfies the same initial conditions, satisfies these initial conditions, initial values. In other words, I started with my little solution. u(x) walks up to and says, hi there. Hi there, and the differential equation looks at it and says, who are you?

You say, oh, I satisfy you and my initial, and then it says what are your initial values? It says, my initial values are u0 and *u0'. And, it said, sorry, but we've got one of ours who satisfies the same initial conditions. We don't need you because the existence and uniqueness theorem says that there can only be one function which does that. And therefore, you must be equal to this guy by the uniqueness theorem.

Okay, we'll talk more about stuff next time, linear equation next time.