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18.03 Differential Equations, Spring 2006
Transcript – Lecture 23

Well, today is the last day on Laplace transform and the first day before we start the rest of the term, which will be spent on the study of systems. I would like to spend it on one more type of input function which, in general, your teachers in other courses will expect you to have had some acquaintance with. It is the kind associated with an impulse, so an input consisted of what is sometimes called a unit impulse. Now, what's an impulse? It covers actually a lot of things. It covers a situation where you withdraw from a bank account. For example, take half your money out of a bank account one day. It also would be modeled the same way.

But the simplest way to understand it the first time through is as an impulse, if you know what an impulse is. If you have a variable force acting over time, and we will assume it is acting along a straight line so I don't have to worry about it being a vector, then the impulse, according to physicists, the physical definition, the impulse of f(t) over some time interval. Let's say the time interval running from a to b is, by definition, the Integral from a to b of [f(t) dt]. Actually, I am going to do the most horrible thing this period. I will assume the force is actually a constant force. So, in that case, I wouldn't even have to bother with the integral at all. If f of t is a constant, let's say capital F, then the impulse is --

Well, that integral is simply the product of the two, the impulse over that time interval is simply F*(b - a). Just the product of those two. The force times the length of time for which it acts. Now, that is what I want to calculate, want to consider in connection with our little mass system. So, once again, I think this is probably the last time you'll see the little spring. Let's bid a tearful farewell to it. There is our little mass on wheels. And let's make it an undamped mass. It has an equilibrium point and all the other little things that go with the picture. And when I apply an impulse, what I mean is applying a constant force to this over a definite time interval.

And that is what I mean by applying an impulse over that time interval. Now, what is the picture of such a thing? Well, the force is only going to be applied, in other words, I am going to push on the mass or pull on the mass with a constant force. With a little electromagnet here, we will assume, there is a pile of iron filings or something inside there. I turn on the electromagnet. It pulls with a constant force just between time zero and time two seconds.

And then I stop. That is going to change the motion of the thing. First it is going to start pulling it toward the thing. And then, when it lets go, it will zoom back and there will be a certain motion after that. What the question is, if I want to solve that problem of the motion of that in terms of the Laplace transform, how am I going to model this force? Well, let's draw a picture of it first.

It starts here. It is zero for t, let's say the force is applied between time zero to time h. And then its force is turned on, it stays constant and then it is turned off. And those vertical lines shouldn't be there. But, since in practice, it takes a tiny bit of time to turn a force on and off. It is, in practice, not unrealistic to suppose that there are approximately vertical lines there. They are slightly slanted but not too much. Now, I want it to be unit impulse. This is the force access and this is the time access. Since the impulse is the area under this curve, if I want that to be one, then if this is h, the height to which I --

In other words, the magnitude of the force must be 1 / h in order that the area be one, in order, in other words, that this integral be one, the area under that curve be one. So the unit impulse looks like that. The narrower it is here, the higher it has to be that way. The bigger the force must be if you want the end result to be a unit impulse. Now, to solve a problem, a typical problem, then, would be a spring.

The mass is traveling on the track. Let's suppose the spring constant is one, so there would be a differential equation. And the right-hand side would be this f of t. Well, let's give it its name, the name I gave it before. Remember, I called the unit box function the thing which was one between zero and h and zero everywhere else. The notation we used for that was u, and then it had a double subscript from the starting point and the finishing point. So oh-- u(oh) (t). This much represents the thing if it only rose to the high one. But if it, instead, rises to the height 1 / h in order to make that area one, I have to multiply it by the factor one over h. Now, if you want to solve this by the Laplace transform.

In other words, see what the motion of that mass is as I apply this unit impulse to it over that time interval. You have to take the Laplace transform, if that is the way we are doing it. Now, the left-hand side is just routine and would involve the initial conditions. The whole interest is taking the Laplace transform of the right-hand side. And that is what I want to do now. The problem is what is the Laplace transform of this guy?

Well, remember, to do everything else, you do everything by writing in terms of the unit step function? This function that we are talking about is 1 / h times what you get by first stepping up to one. That is the unit step function, which goes up by one and tries to stay at one ever after. And then, when it gets to h, it has got to step down. Well, the way you make it step down is by subtracting off the function, which is the unit step function but where the step takes place, not at time zero but at time h.

In other words, I translate the unit step function of course with, I don't think I have to draw that picture again. The unit step function looks like zing. And if you translate it to the right by h it looks like zing. And then make it negative to subtract it off. And what you will get is this box function. So we want to take the Laplace transform of this thing. Well, let's assume, for the sake of argument that you didn't remember. Well, you had to use the formula at 2:00 AM this morning and, therefore, you do remember it. [LAUGHTER] So I don't have to recopy the formula onto the board. Maybe if there is room there. All right, let's put it up there. It says that u(t - a) times f, any f, so let's call it g so you won't confuse it with this particular one, times g translated.

If you translate a function from t, if you translate it to the right by a then its Laplace transform is e^(-as) times whatever the old Laplace transform was, g(s). Multiply by an exponential on the right. On the left that corresponds to translation. Except you must remember to put in that factor u for a secret reason which I spent half of Wednesday explaining. What do we have here? The L(u(t)), that is easy. That is simply 1 / s. The Laplace transform of this other guy we get from the formula. It is basically one over s. No, the L(u(t). But because it has been translated to the right by h, I have to multiply it by that factor e^(-hs).

That is the answer. And, if you want to solve problems, this is what you would feed into the equation. And you would calculate and calculate and calculate it. But that is not what I want to do now because that was Wednesday and this is Friday. You have the right to expect something new. Here is what I am going to do new. I am going to let h go to zero. As h --> 0, this function gets narrower and narrower, but it also has to get higher and higher because its area has to stay one. What I am interested in, first of all, is what happens to the Laplace transform as h goes to zero. In other words, what is the limit, as h goes to zero of --

Well, what is that function? (1 - e^(-hs)) / hs. Well, this is an 18.01 problem, an ordinary calculus problem, but let's do it nicely. You see, the nice way to do it is to make a substitution. We will change hs --> u because it is occurring as a unit in both cases. This is going to be the same as the limit as u --> 0.

I think there are too many u's here already. I cannot use u, you cannot use t, v is velocity, w is wavefunction. There is no letter. All right, u. It is (1 - e^(-u)) / u. So what is the answer? Well, either you know the answer or you replace this by, say, the first couple of terms of the Taylor series. But I think most of you would use L'Hopital's rule, so let's do that. The derivative of the top is zero here. The derivative by the chain rule of e to the negative u is (e^(-u))' = -1*e^(-u). And that minus one cancels that minus. So the derivative of the top is simply e to the negative u and the derivative of the bottom is one. So, as u --> 0, that limit is one.

Interesting. Let's draw a picture this way. I will draw it schematically. Up here is the function 1 / h u0 h(t), our box function, except it has the height 1 / h instead of the height one. We have just calculated that its Laplace transform is that funny thing, (1 - e^(-hs)) / hs. That is the top line. All this is completely kosher, but now I am going to let h go to zero. And the question is what do we get now? Well, I just calculated for you that this thing approaches one, has the limit one. And now, let's fill in the picture.

What does this thing approach? Well, it approaches a function which is zero everywhere. As h approaches zero, this green box turns into a box which is zero everywhere except at zero. And there, it is infinitely high. So, keep going up. Now, of course, that is not a function. People call it a function but it isn't. Mathematicians call it a generalized function, but that is not a function either. It is just a way of making you feel comfortable by talking about something which isn't really a function. It was given the name, introduced formally into mathematics by a physicist, Dirac.

And he, looking ahead to the future, did what many people do who introduce something into the literature, a formula or a function or something which they think is going to be important. They never name it directly after themselves, but they always use as the symbol for it the first letter of their name. I cannot tell you how often that has happened. Maybe even Euler called e for that reason, although he claims it was in Latin because it has to do with exponentials. Well, luckily his name began with an E, too. That is Paul Dirac's delta function. I won't dignify it by the name function by writing that out, by putting the world function here, too, but it is called the delta function.

From this point on, the entire rest of the lecture has a slight fictional element. The entire rest of the lecture is in figurative quotation marks, so you are not entirely responsible for anything I say. This is a non-function, but you put it in there and call it a function. And you naturally want to complete, if it's a function then it must have a Laplace transform, even though it doesn't, so the diagram is completed that way.

And its Laplace transform is declared to be one. So let's start listing the properties of this weird thing. The delta function, its Laplace transform is one. Now, one of the things is we have not yet expressed the fact that it is a unit impulse. In other words, since the areas of all of these boxes, they all have areas one as they are shrunk this way they get higher that way. By convention, one says that the area under the orange curve also remains one in the limit.

Now, how am I going to express that? Well, it is done by the following formula that the integral, the total impulse of the delta function should be one. Now, where do I integrate? Well, from any place that it is zero to any place that it is zero on the other side of that vertical line. But, in order to avoid controversy, people integrate all the way from negative infinity to infinity since it doesn't hurt. Does it? It is zero practically all the time.

This is the function whose Laplace transform is one. Its integral from minus infinity to infinity is one. How else can we calculate for it? Well, I would like to calculate its convolution. Here is f(t). What happens if I convolute it with the delta function? Well, if you go back to the definition of the convolution, you know, it is that funny integral, you are going to do a lot of head scratching because it is not really all that clear how to integrate with the delta function. Instead of doing that let's assume that it follows the laws of the Laplace transform. In that case, its Laplace transform would be what? Well, the whole thing of a convolution is that the Laplace transform of the convolution is the product of the two separate Laplace transforms.

So that is going to be F(s) times the Laplace transform of the delta function, which is one. Now, what must this thing be? Well, there is some ambiguity as to what it is for negative values of t. But if we, by brute force, decide for negative values of t it is going to have the value zero, that is the way we make things unique. In fact, why don't we make f(t) unique that way to start with? This is a function now that is allowed to do anything it wants on the right-hand side of zero starting at zero, but on the left-hand side of zero it is wiped away and must be zero. This is a definite thing now. Its convolution is this. And the inverse Laplace transform is --

The answer, in other words, is the same thing as what u(t) f(t) would be. It's the same thing, F(s). And so, the conclusion is that these are equal, since they must be unique. They have been made unique by making them zero for t negative. In other words, apply to a function, well, I won't recopy it. But the point is that delta t, for the convolution operation, is acting like an identity.

If I multiply, in the sense of convolution, it is a peculiar operation. But algebraically, it has a lot of the properties of multiplication. It is communitive. It is linear in both factors. In other words, it is almost anything you would want with multiplication. It has an identity element, identity function. And the identity function is the Dirac delta function. Anything else here? Yeah, I will throw in one more thing.

It would just require one more phony argument, which I won't bother giving you, but it is not totally implausible. After all, u(t), the unit step function is not differentiable, is not a differentiable function. It looks like this. Here its derivative is zero, here its derivative is zero, and in this class it is not even defined in between. But, I don't care, I will make it go straight up. The question is what's its derivative?

Well, zero here, zero there and infinity at zero, so it must be the delta function. That has exactly the right properties. So the same people who will tell you this will tell you that also. And, in fact, when you use it to solve differential equations it acts as if that is true. I think I have given you an example on your homework. Let me now show you a typical example of the way the Dirac delta function would be used to solve a problem.

Let's go back to our little spring, since it is the easiest thing. You are familiar with it from a physical point of view, and it is the easiest thing to illustrate on. We have our spring mass system. Where is it? Is it on the board? Up there. That one. And the differential equation we are going to solve is $y'' + y$ equals -- And now, I am going to assume that the spring is kicked with impulse a. I am not going to kick it at time $t = 0$, since that would get us into slight technical difficulties. Anyway, it is more fun to kick it at time pi / 2. The thing is, what is happening? Well, we have got to have initial conditions.

The initial conditions are going to be, let's start at time zero. We will start it at the position one. So I take my spring, I drag it to the position one, I take the little mass there and then let it go. And so it starts going birr. But right when it gets to the equilibrium point I give it a, "cha!" with unit impulse. I started it from rest. Those will be the initial conditions. And I want to say that I kicked it, not with unit impulse, but with the impulse a. Bigger. And I did that at time pi over two. So how are we going to say that? Well, kick it means delivered that impulse over an extremely short time interval, but in such a way kicked it sufficiently hard that the total impulse was a. The way to say that is kick it with the Dirac delta function.

Translate it to the point time pi / 2. Not at zero any longer. t - pi / 2. But that would kick it with a unit impulse. I want it to kick it with the impulse a, so I will just multiply that by the constant factor a. Let's put this over here. $y(0) = 1$, that's the starting value. Now we have a problem. The only thing new in solving this with the Laplace transform is I have this funny right-hand side. But it corresponds to a physical situation. Let's do it. You take the Laplace transform of both sides of the equation. Remember how to do that? You have to take account of the initial conditions. The Laplace transform of the second derivative is you multiply by s^2, and then you have to subtract. You have to use these initial conditions.

This one won't give you anything, but the first one means I have to subtract 1*s. That is the $L(y'')$. $L(y) = Y$. And how about the Laplace transform of the right-hand side. Well, we will have the constant factor a because the Laplace transform is linear. And now, the delta function would have the transform one.

But when I translate it, pi over two, that means I have to use that formula. Translate it by pi over two means take the one that it would have been otherwise and multiply it by e, that exponential factor. It would be $e^{(-\pi / 2)}$, that is the A times s times one, which would be the g(s), the Laplace transform or the delta function before it had been translated. But I don't have to put that in because it's one.

I am multiplying by one. And to do everything now is routine. Solve for the Laplace transform. Well, what is it? It is y is equal to. I put the s on the other side. That makes the right-hand side the sum of two terms. And I divide by the coefficient of y, which is s^2 + 1. The s is over on the right-hand side and it is divided by s squared plus one. And the other factor is there, too. And it, too, is divided by s squared plus one.

Now, we take the inverse Laplace transform of those two terms and add them up. What will we get? Well, y is equal to, the L^(-1) s / (s^1 + 1) = cos(t). Now, for this thing we will have to use our formula. If this weren't here, the L^(-1) A / (s^2 + 1) would be what? Well, it would be a times the A sin(t).

In other words, if this is the g(s) then the function on the left would be basically A sin(t). But because it has been multiplied by that exponential factor, e^(-as) where a = pi / 2, the left-hand side has to be changed from A sin(t) to what it would be with the translated form. So the rest of it is u(t - pi/2), because a is pi over two, times what it would have been just from the factor g(s) itself. In other words, A sin(t - pi/2). I am applying that formula, but I am applying it in that direction. I started with this, and I want to recover the left-hand side. And that is what it must look like. The A, of course, just gets dragged along for the free ride.

Now, as I emphasized to you last time, and I hope you did on your homework that you handed in, you mustn't leave it in that form. You have to make cases because people will expect you to tell them what the meaning of this is. Now, if t is less than pi over two, this is zero. And, therefore, that term does not exist. So the first part of it is just the cos(t) term if t lies between zero and pi over two.

If t > pi/2 then this factor is one. It's the unit step function. And I, therefore, must add in this term. Now, what is that term? What is the sin(t - pi/2)? The sin(t) looks like that. The sine of t, if I translate it, looks like this. If I translate it by pi over two. And let's finish it up, the pi that was over here moved into position. That curve is the curve -cos(t).

And so the answer is if t > pi / 2, it is cos(t) - Acos(t). Or, in other words, it is (1 - A) cos(t). Now, do those match up? They have always got to match up, or you have made a mistake. You always have to get a continuous function when you have just discontinuities. Do we get a continuous function? Yeah, when t = pi / 2 the value here is zero. The value of this is also zero at pi over two.

There is no conflict in the values. Values doesn't suddenly jump. The function is continuous. It is not differential but it is continuous. Well, what function does that look like? There are cases. It starts out life as the function cos(t). So it gets to here. And at t = pi / 2, the mass gets kicked and that changes the function. Now, what are the values?

Well, if A > 1 this is a negative number and it therefore becomes the function -cos(t). Now, negative cosine t looks like this, the blue guy. Negative cosine t is a function that looks like this. So it goes from here, it reverses direction, the mass reverses direction from what you thought it was going to do. And it does that because A is so large that that impulse was enough to make it reverse direction. Of course it might only do this, but this is what will happen if A is bigger than one. This will be A, which is a lot bigger than one. If it's not so much bigger than one it might look like that. So A is just bigger than one. How's that?

Well, what if A is less than one? Well, in that case it stays positive. If A is less than one, this is now still a positive number. And, therefore, the cosine continues on its merry way. The only thing is it might be a little more sluggish or it might be very peppy and do that. Let's just go that far. This will be the case A less than one. Well, of course, the most interesting case is what happens if A = 1?

The porridge is exactly just right, I think that's the phrase. Too hot. Too cold. Just right. When A is equal to one, it is zero. It starts out as cos(t). When it gets to t, it continues on ever after as the function zero. I have a visual aid for the only time this term. It didn't work at all. I mean, on the other hand, the last hour, the people who worked it were not intrinsically baseball players, so we will use the equation of the pendulum instead. That is a lot easier than mass spring. This is a pendulum. It is undamped because I declare it to be and it swings back and forth.

And here I am releasing it. The variable is not x or y but theta, the angle through. Here theta is one, let's say. That's about one radian. It starts there and swings back and forth. It is not damped, so it never loses amplitude, particularly if I swish it, if I move my hand a little bit. I want someone who knows how to bat a baseball. That was the problem last hour. Two people. One to release it. I will stand up and try to hold it here. Somebody releases it. And then somebody who has to be very skillful should apply a unit impulse of exactly one when it gets to the equilibrium point. So who can do that? Who can play baseball here? Come on. Somebody elected?

All right. Come on. [APPLAUSE] Somebody release it, too. Somebody tall to handle it all. I think that will be me. Just hold it at what you would take to be one radian. He releases it. When it gets to the bottom, you will have to get way down, and maybe on this side. Are you a lefty or a righty? Rightly. Okay. Bat it what part. Give it a good swat. I will stand up higher. Help. I'm not very stable. [APPLAUSE] A trial run.

Again. Okay. A little further out. First of all, you have to see where it's going. Why don't you stand, oh, you bat rightly. That's right. Okay. Let's try it again. Strike one. It's okay. It's the beginning of the baseball season. One more. The Red Sox are having trouble, too. Not bad. [APPLAUSE] If he had hit even harder it would have reversed direction and gone that way. If you hadn't hit it quite as hard it would have continued on, still at cos(t), but with less amplitude. But if you hit it exactly right --

It is fun to try to do. Toomre in our department is a master at this, but he has been practicing for years. He can take a little mallet and go blunk, and it stops absolutely dead. It is unbelievable. I should have had him give the lecture. Now, I would like to do something truly serious. Here, I guess. Because there is a certain amount of engineering lingo you have to learn. It is used by almost everybody. Not architects and biologists probably quite yet, but anybody that uses the Laplace transform will use these words in connection with it. I really think, since it is such a widespread technique, that these are things you should know.

Anyway, it will be easy. It is just the enrichment of your vocabulary. It is always fun to learn new vocabulary words. So, let's just consider a general second order equation. By the way, all this applies to higher order equations, too. It applies to systems. The same words are used, but let's use something that you know. Here is a system. It could be a spring mass dashpot system. It could be an RLC circuit.

Or that pendulum, a damped pendulum, anything that is modeled by that differential equation with constant coefficients, second-order. This is the input. The input can be any kind of a function. Exponential functions, sine, cosine. It could be a Dirac delta function. It could be a sum of these things. It could be a Fourier series. Anything of the sort of stuff we have been talking about throughout the last few weeks.

And let's have simple initial conditions so that doesn't louse things up, the simplest possible ones. The mass starts at the equilibrium point from rest. Of course, it doesn't stay that way because there is an input that is asking it to move along. Now all I want to do is solve this in general with a Laplace transform. If I do it in general, that is always easier than doing it in particular since you don't ever have to do any calculations.

It is s^2 Y. There are no other terms here because the initial conditions are zero. This part will be a sY. Again, no other terms because the initial conditions are zero. Plus bY. And all that is equal to whatever the Laplace transform is of the right-hand side. So it is F(s). Next step. Boy, this is an easy problem. You solve for Y. Well, Y = F(s) * (1 / (s^2 + as + b)). Now, what is that? The next step now is to figure out what the answer to the problem is, what's the Y(t)?

Well, you do that by taking the inverse Laplace transform. But because these are general functions, I don't have to write down any specific answer. The only thing is to use the convolution because this is the product of two functions of s. The inverse transform will be the convolution of their respective things. The answer is going to be the convolution of F(t), the input function in other words, convoluted with the inverse Laplace transform of that thing.

Now, we have to have a name for that, and those are the two words I want to introduce you to because they are used everywhere. The function, on the right-hand side, this function 1 / (s^2 + as + b), notice it only depends upon the left-hand side of the differential equation, on the damping constant. The spring constant if you are thinking of a mass spring dashpot system. So this depends only on the system, not on what input is going into it. And it is called the transfer function. Is usually called capital W(s), sometimes it is capital H(s), there are different things, but it is always called the transfer function.

What we are interested in putting here its inverse Laplace transform. Well, I will call that W(t) to go with the capital W(s) by the usual notation. Its inverse Laplace transform, well, I cannot calculate that. I will just give it a name, W of t. And that is called the weight function of the system. This is the transfer function of the system, so put in "of the system" if you are taking notes.

And so the answer is that always the solution is the convolution to this differential equation that we have been solving for the last three or four weeks. It is the convolution of that. And, therefore, the solution is expressed as a definite integral of the function of the input on the right-hand side, what is forcing the equation, times this magic function but flipped and translated by t. That says du for you guys over there. In other words, the solution to the differential equation is presented as a definite integral. Marvelous. And the only thing is the definite integral involves this funny function W(t). To understand why that is the solution, you have to understand what W of t is. Well, formally, of course, it's that.

But what does it really mean? The problem is what is W(t) really? Not just formally, but what does it really mean? I mean, is it real? I think the simplest way of thinking of it, once you know about the delta function is just to think of this differential equation $y'' + ay' + b$. Except use as the input the Dirac delta function. In other words, we are kicking the mass. The mass starts at rest, so the initial conditions are going to be what they were before. $y(0)$, $y'(0)$. Both zero. The mass starts at rest from the equilibrium position, and it is kicked in the positive direction, I guess that's this way, with unit impulse.

At time zero with unit impulse. In other words, kick it just hard enough so you impart a unit impulse. So that situation is modeled by this differential equation. The kick at time zero is modeled by this input, the Dirac delta function. And now, what happens if I solve it? Well, you see, everything in the solution is the same. The left stays the same, but on the right-hand side I should have not $f(s)$ here. Since this is the delta function, I should have one. What I get is, on the left-hand side, $s^2 Y + asY + bY$ equals, for the Laplace transform of the right-hand side is simply one. And, therefore, Y is what?

Y is one over exactly the transform function. And therefore its inverse Laplace transform is that weight function. That is the simplest interpretation I know of what this magic weight function is, which gives the solution to all the differential equations, no matter what the input is. The weight function is the response of the system at rest to a sharp kick at time zero with unit impulse. And read the notes because they will explain to you why this could be thought of as the superposition of a lot of sharp kicks times zero a little later. Kick, kick, kick, kick. And that's what makes the solution. Next time we start systems.