MIT OpenCourseWare
http://ocw.mit.edu

18.03 Differential Equations, Spring 2006
Transcript – Lecture 32

First of all, the way a nonlinear autonomous system looks, you have had some practice with it by now. This is nonlinear. The right-hand side are no longer simple combinations ax + by. Nonlinear and autonomous, these are function just of x and y. There is no t on the right-hand side. Now, most of today will be geometric. The way to get a geometric picture of that is first by constructing the velocity field whose components are the functions f and g. This is a velocity field that gives a picture of the system and has solutions.

The solutions to the system, from the point of view of functions, they would look like pairs of functions, x(t), y(t). But, from the point of view of geometry, when you plot them as parametric equations, they are called trajectories of the field F, which simply means that they are curves everywhere having the right velocity. So a typical curve would look like -- There is a trajectory. And we know it is a trajectory because at each point the vector on it has, of course, the right direction, the tangent direction, but more than that, it has the right velocity. So here, for example, the point is traveling more slowly. Here it is traveling more rapidly because the velocity vector is bigger, longer.

So this is a picture of a typical trajectory. The only other things that I should mention are the critical points. If you have worked the problems for this week, the first couple of problems, you have already seen the significance of the critical points. Well, from Monday's lecture you know from the point of view of solutions they are constant solutions. From the point of view of the field they are where the field is zero. There is no velocity vector, in other words. The velocity vector is zero. And, therefore, a point being there has no reason to go anywhere else. And, spelling it out, it's where the partial derivatives, where the values of the functions on the right-hand side, which give the two components, the i and j components of the field, where they are zero.

That is all I will need by way of a recall today. I don't think I will need anything else. The topic for today is another kind of behavior that you have not yet observed at the computer screen, unless you have worked ahead, and that is there are trajectories which go along to infinity or end up at a critical point. They are the critical points that just sit there all the time. But there is a third type of behavior that a trajectory can have where it neither sits for all time nor goes off for all time. Instead, it repeats itself. Such a thing is called a closed trajectory. What does it look like?

Well, it is a closed curve in the plane that at every point, it is a trajectory, i.e., the arrows at each point, let's say it is traced in the clockwise direction. And so the arrows of the field will go like this. Here it is going slowly, here it is very slow and here it picks up a little speed again and so on. Now, for such a trajectory what is happening? Well, it goes around in finite time and then repeats itself. It just goes round and round forever if you land on that trajectory. It represents a system that returns to its original state periodically. It represents periodic behavior of the system.

Now, we have seen one example of that, a simple example where this simple system, $x' = y$, $y' = -x$. We could write down the solutions to that directly, but if you want to do eigenvalues and eigenvectors the matrix will look like this. The equation will be $\lambda^2 + 0\lambda + 1 = 0$, so the eigenvalues will be plus or minus i.

In fact, from then on you could work out in the usual ways the eigenvectors, complex eigenvectors and separate them. But, look, you can avoid all that just by writing down the solution. The solutions are sines and cosines. One basic solution will be $x = \cos(t)$, in which case what is y? Well, y is the derivative of that. That will be $y = -\sin(t)$. Another basic solution, we will start with $x = \sin(t)$.

In which case $y = \cos(t)$, its derivative. Now, if you do that, what do these things look like? Well, either of these two basic solutions looks like a circle, not traced in the usual way but in the opposite way. For example, when t is equal to zero it starts at the point one, zero. Now, if the minus sign were not there this would be $x = \cos(t)$, $y = \sin(t)$, which is the usual counterclockwise circle. But if I change $\sin(t) \rightarrow -\sin(t)$ it is going around the other way. So this circle is traced that way. And this is a family of circles, according to the values of c_1 and c_2, concentric, all of which go around clockwise.

So those are closed trajectories. Those are the solutions. They are trajectories of the vector field. They are closed. They come around and they repeat in finite time. Now, these are no good. These are the kind I am not interested in. These are commonplace, and we are interested in good stuff today. And the good stuff we are interested in is limit cycles. A limit cycle is a closed trajectory with a couple of extra hypotheses. It is a closed trajectory, just like those guys, but it has something they don't have, namely, it is king of the roost. They have to be isolated, no other guys nearby. And they also have to be stable. See, the problem here is that none of these stands out from any of the others.

In other words, there must be, isolated means, no others nearby. That is just what goes wrong here. Arbitrarily close to each of these circles is yet another circle doing exactly the same thing. That means that there are some that are only of mild interest. What is much more interesting is to find a cycle where there is nothing nearby. Something, therefore, that looks like this. Here is our pink guy. Let's make this one go counterclockwise. Here is a limit cycle, it seems to be.

And now what do nearby guys do? Well, they should approach it. Somebody here like that does this, spirals in and gets ever and every closer to that thing. Now, it can never join it because, if it joined it at the joining point, I would have two solutions going through this point. And that is illegal. All it can do is get arbitrarily close. On the computer screen it will look as if it joins it but, of course, it cannot. It is just the resolution, the pixels. Not enough pixels. The resolution isn't good enough. And the ones that start further away will take longer to find their way to the limit cycle and they will always stay outside of the earlier guys, but they will get arbitrarily close, too. How about inside?

Inside, well, it starts somewhere and does the same thing. It starts here and will try to join the limit cycle. That is what I mean by stability. Stability means that nearby guys, the guys that start somewhere else eventually approach the limit cycle, regardless of whether they start from the outside or start from the inside. So that is stable. An unstable limit cycle -- But I am not calling it a limit cycle if it is unstable. I am just calling it a closed trajectory, but let's draw one which is unstable. Here is the way we will look if it is unstable. Guys that start nearby will be repelled, driven somewhere else. Or, if they start here, they will go away from the thing instead of going toward it. This is unstable. And I don't call it a limit cycle. It is just a closed trajectory.

Cycle because it cycles round and round. Limit because it is the limit of the nearby curves. The other case where it is unstable is not the limit. Of course, you could have a case also where the curves outside spiral in toward it but the ones inside are repelled and do this. That would be called semi-stable. And you can make up all sorts of cases. And I think I, at one point, drew them in the notes, but I am not going to. The only interesting one, of permanent importance that people study, are the actual limit cycles. No, it was the stable closed trajectories. Notice, by the way, a closed trajectory is always a simple curve. Remember what that means from 18.02?

Simple means it doesn't cross itself. Why doesn't it cross itself? It cannot cross itself because, if it tried to, what is wrong with that point? At that point which way does the vector field go, that way or that way? Why the interest of limit cycles? Well, because there are systems in nature in which just this type of behavior, they have a certain periodic motion. And, if you disturb it, gradually it returns to its original periodic state. A simple example is breathing. Now I have made you all self-conscious. All of you are breathing. If you are here you are breathing. At what rate are you breathing? Well, you are unaware of it, of course, except now. If you are sitting here listening.

There is a certain temperature and a certain air circulation in the room. You are not thinking of anything, certainly not of the lecture, and the lecture is not unduly exciting, you will breathe at a certain steady rate which is a little different for every person but that is your rate. Now, you can artificially change that. You could say now I am going to breathe faster. And indeed you can. But, as soon as you stop being aware of what you are doing, the levels of various hormones and carbon dioxide in your bloodstream and so on will return your breathing to its natural state. In other words, that system of your breathing, which is controlled by various chemicals and hormones in the body, is exhibiting exactly this type of behavior. It has a certain regular periodic motion as a system.

And, if disturbed, if artificially you set it out somewhere else, it will gradually return to its original state. Now, of course, if I am running it will be different. Sure. If you are running you breathe faster, but that is because the parameters in the system, the a's and the b's in the equation, the $f(x, y)$ and $g(x, y)$, the parameters in those functions will be set at different levels. You will have different hormones, a different of carbon dioxide and so on.

Now, I am not saying that breathing is modeled by a limit cycle. It is the sort of thing which one might look for a limit cycle. That is, of course, a question for biologists. And, in general, any type of periodic behavior in nature, people try to see if there is some system of differential equations which governs it in which perhaps there is a limit cycle, which contains a limit cycle. Well, what are the problems? In a sense, limit cycles are easy to lecture about because so little is known about them. At the end of the period, if I have time, I will show you that the simplest possible question you could ask, the answer to it is totally known after 120 years of steady trying.

But let's first talk about what sorts of problems people address with limit cycles. First of all is the existence problem. If I give you a system, you know, the right-hand side is $x^2 + 2y^3 - 3xy$, and the g is something similar. I say does this have limit cycles? Well, you know how to find its critical points. But how do you find out if it has limit cycles? The answer to that is nobody has any idea. This problem, in general, there are not much in the way of methods. Not much.

Not much is known. There is one theorem that you will find in the notes, a simple theorem called the Poincare-Bendixson theorem which, for about 60 or 70 years was about the only result known which enabled people to find limit cycles. Nowadays the theorem is used relatively little because people try to find limit cycles by computer. Now, the difficulty is you have to know where to look for them.

In other words, the computer screen shows that much and you set the axes and it doesn't show any limit cycles. That doesn't mean there are not any. That means they are over there, or it means there is a big one like there. And you are looking in the middle of it and don't see it. So, in general, people don't look for limit cycles unless the physical system that gave rise to the pair of differential equations suggests that there is something repetitive going on like breathing.

And, if it tells you that, then it often gives you approximate values of the parameters and the variables so you know where to look. Basically this is done by computer search guided by the physical problem. Therefore, I cannot say much more about it today. Instead I am going to focus my attention on nonexistence. When can you be sure that a system will not have any limit cycles? And there are two theorems. One, again, due to Bendixson who was a Swedish mathematician who lived around 1900 or so. There is a criterion due to Bendixson.

And there is one involving critical points. And I would like to describe both of them for you today. First of all, Bendixson's criterion. It is very simply stated and has a marvelous proof, which I am going to give you. We have D as a region of the plane. And what Bendixson's criterion tells you to do is take your vector field and calculate its divergence. We are set back in 1802, and this proof is going to be straight 18.02. You will enjoy it. Calculate the divergence. Now, I am talking about the two-dimensional divergence.

Remember that is fx, the partial of f with respect to x, plus the partial of the g, the j component with respect to y. And assume that that is a continuous function. It always will be with us. Practically all the examples I will give you f and g will be simple polynomials. They are smooth, continuous and nice and behave as you want. And you calculate that and assume -- Suppose, in other words, that the divergence of f, I need more room. The hypothesis is that the divergence of f is not zero in that region D. It is never zero. It is not zero at any point in that region. The conclusion is that there are no limit cycles in the region. If it is not zero in D, there are no limit cycles.

In fact, there are not even any closed trajectories. You couldn't even have those bunch of concentric circles, so there are no closed trajectories of the original system whose divergence you calculated. There are no closed trajectories in D. For example, let me give you a simple example to put a little flesh on it. Let's see. What do I have? I prepared an example. Here is a simple nonlinear system, $x' = x^3 + y^3$. And $y' = 3x + y^3 + 2y$.

Does this system have limit cycles? Well, even to calculate its critical points would be a little task, but we can easily answer the question as to whether it has limit cycles or not by Bendixson's criterion. Let's calculate the divergence. The divergence of the vector field whose components are these two functions is, well, $3x^2$, it's the partial of the first guy with respect to x plus the partial of the second guy with respect to y, which is $3y^2 + 2$.

Now, can that be zero anywhere in the x,y-plane? No, because it is the sum of these two squares. This much of it could be zero only at the origin, but that plus two eliminates even that. This is always positive in the entire x,y-plane. Here my domain is the whole x,y-plane and, therefore, the conclusion is that there are no closed trajectories in the x,y-plane, anywhere. And we have done that with just a couple of lines of calculation and nothing further required. No computer search. In fact, no computer search could ever proof this. It would be impossible because, no matter where you look, there is always some other place to look.

This is an example where a couple lines of mathematics dispose of the matter far more effectively than a million dollars worth of calculation. Well, where does Bendixson's theorem come from? Yes, Bendixson's theorem comes from 18.02. And I am giving it to you both to recall a little bit of 18.02 to you. Because it is about the first example in the course that we have had of an indirect argument. And indirect arguments are something you have to slowly get used to. I am going to give you an indirect proof. Remember what that is? You assume the contrary and you show it leads to a contradiction. What would assuming the contrary be?

Contrary would be I will assume the divergence is not zero, but I will suppose there is a closed trajectory. Suppose there is a closed trajectory that exists. Let's draw a picture of it. And let's say it goes around this way. There is a closed trajectory for our system. Let's call the curve C. And I am going to call the inside of it R, the way one often does in 18.02. D is all this region out here, in which everything is taking place.

This is to exist in D. Now, what I am going to do is calculate a line integral around that curve. A line integral of this vector field. Now, there are two things you can calculate. One of the line integrals, I will put in a few of the vectors here. The vectors I know are pointing this way because that is the direction in which the curve is being traversed in order to make it a trajectory. Those are a few of the typical vectors in the field. I am going to calculate the line integral around that curve in the positive sense. In other words, not in the direction of the salmon-colored arrow, but in the normal sense in which you calculate it using Green's theorem, for example.

The positive sense means the one which keeps the region, the inside on your left, as you walk around like that the region stays on your left. That is the positive sense. That is the sense in which I am integrating. I am going to use Green's theorem, but the integral that I am going to calculate is not the work integral. I am going to calculate instead the flux integral, the integral that represents the flux of F across C.

Now, what is that integral? Well, at each point, you station a little ant and the ant reports the outward flow rate across that point which is F dotted with the normal vector. I will put in a few normal vectors just to remind you. The normal vectors look like little unit vectors pointing perpendicularly outwards everywhere. These are the n's. F dotted with the unit normal vector, and that is added up around the curve. This quantity gives me the flux of the field across C. Now, we are going to calculate that by Green's theorem. But, before we calculate it by Green's theorem, we are going to psych it out. What is it? What is the value of that integral?

Well, since I am asking you to do it in your head there can only be one possible answer. It is zero. Why is that integral zero? Well, because at each point the field vector, the velocity vector is perpendicular to the normal vector. Why? The normal vector points perpendicularly to the curve but the field vector always is tangent to the curve because this curve is a trajectory. It is always supposed to be going in the direction given by that white field vector. Do you follow? A trajectory means that it is always tangent to the field vector and, therefore, always perpendicular to the normal vector. This is zero since F dot n is always zero.

Everywhere on the curve, F dot n has to be zero. There is no flux of this field across the curve because the field is always in the same direction as the curve, never perpendicular to it. It has no components perpendicular to it. Good. Now let's do it the hard way. Let's use Green's theorem. Green's theorem says that the flux across C should be equal to the double integral over that region of the divergence of F.

It's like Gauss theorem in two dimensions, this version of it. Divergence of F, that is a function, I double integrate it over the region, and then that is dx / dy, or let's say da because you might want to do it in polar coordinates. And, on the problem set, you certainly will want to do it in polar coordinates, I think. All right. How much is that? Well, we haven't yet used the hypothesis. All we have done is set up the problem. Now, the hypothesis was that the divergence is never zero anywhere in D. Therefore, the divergence is never zero anywhere in R. What I say is the divergence is either greater than zero everywhere in R.

Or less than zero everywhere in R. But it cannot be sometimes positive and sometimes negative. Why not? In other words, I say it is not possible the divergence here is one and here is minus two. That is not possible because, if I drew a line from this point to that, along that line the divergence would start positive and end up negative. And, therefore, have to be zero some time in between. It's because it is a continuous function. It is a continuous function. I am assuming that. And, therefore, if it sometimes positive and sometimes negative it has to be zero in between. You cannot get continuously from plus one to minus two without passing through zero.

The reason for this is, since the divergence is never zero in R it therefore must always stay positive or always stay negative. Now, if it always stays positive, the conclusion is then this double integral must be positive. Therefore, this double integral is either greater than zero. That is if the divergence is always positive. Or, it is less than zero if the divergence is always negative. But the one thing it cannot be is not zero. Well, the left-hand side, Green's theorem is supposed to be true. Green's theorem is our bedrock. 18.02 would crumble without that so it must be true. One way of calculating the left-hand side gives us zero.

If we calculate the right-hand side it is not zero. That is called the contradiction. Where did the contradiction arise from? It arose from the fact that I supposed that there was a closed trajectory in that region. The conclusion is there cannot be any closed trajectory of that region because it leads to a contradiction via Green's theorem. Let me see if I can give you some of the argument for the other, well, let's at least state the other criterion I wanted to give you.

Suppose, for example, we use this system, x prime equals -- Does this have limit cycles? Does that have limit cycles? Let's Bendixson it. We will calculate the divergence of a vector field. It is 2x from the top function. The partial with respect to x is 2x. The second function with respect to y is negative 2y. That certainly could be zero. In fact, this is zero along the entire line x equals y. Its divergence is zero here along that whole line. The best I could conclude was, I could conclude that there is no limit cycle like this and there is no limit cycle like this, but there is nothing so far that says a limit cycle could not cross that because that would not violate Bendixson's theorem.

In other words, any domain that contained part of this line, the divergence would be zero along that line. And, therefore, I could conclude nothing. I could have limit cycles that cross that line, as long as they included a piece of that line in them. The answer is I cannot make a conclusion. Well, that is because I am using the wrong criterion. Let's instead use the critical point criterion.

Now, I am going to say that it makes a nice positive statement but nobody ever uses it this way. Nonetheless, let's first state it positively, even though that is not the way to use it. The positive statement will be, once again, we have our region D and we have a region of the xy plane and we have our C, a closed trajectory in it. A closed trajectory of what? Of our system. And that is supposed to be in D.

The critical point criterion says something very simple. If you have that situation it says that inside that closed trajectory there must be a critical point somewhere. It says that inside C is a critical point. Now, this won't help us with the existence problem. This won't help us find a closed trajectory. We will take our system and say it has a critical point here and a critical point there. Does it have a closed trajectory?

Well, all I know is the closed trajectory, if it exists, will have to go around one or more of those critical points. But I don't know where. It is not going to go around it like this. It might go around it like this. And my computer search won't find it because it is looking at too small a part of the screen. It doesn't work that way. It works negatively by contraposition. Do you know what the contrapositive is?

You will at least learn that. A implies B says the same thing as not B implies not A. They are different statements but they are equivalent to each other. If you prove one you prove the other. What would be the contrapositive here? If you have a closed trajectory inside is a critical point. The theorem is used this way. If D has no critical points, it has no closed trajectories and therefore has no limit cycle.

Because, if it did have a closed trajectory, inside it would be a critical point. But I said B had no critical point. That enables us to dispose of this system that Bendixson could not handle at all. We can dispose of this system immediately. Namely, what is it? Where are its critical points? Well, where is that zero? $x^2 + y^2 = 1$, plus one is never zero. This is positive. Or, worse, zero. And then I add the one to it and it is not zero anymore. This has no zeros and, therefore, it does not matter that this one has a lot of zeros. It makes no difference. It has no critical points. It has none, therefore, no limit cycles.

Now, I desperately wanted to give you the proof of this. It is clearly impossible in the time remaining. The proof requires a little time. I haven't decided what to do about that. It might leak over until Friday's lecture. Instead, I will finish by telling you a story. How is that? And along side of it was little y prime. I am not going to continue on with the letters of the alphabet. I will prime the earlier one. This has a total of 12 parameters in it. But, in fact, if you change variables you can get rid of all the linear terms. The important part of it is only the quadratic terms in the beginning.

This sort of thing is called a quadratic system. After you have departed from linear systems, it is the simplest kind there is. And the predictor-prey, the robin-earthworm example I gave you is of a typical quadratic system and exhibits typical nonlinear quadratic system behavior. Now, the problem is the following. A, b, c, d, e, f and so on, those are just real numbers, parameters, so I am allowed to give them any values I want. And the problem that has bothered people since 1880 when it was first proposed is how many limit cycles can a quadratic system have?

After 120 years this problem is totally unsolved, and the mathematicians of the world who are interested in it cannot even agree with each other on what the right conjecture is. But let me tell you a little bit of its history. There were attempts to solve it in the 20 or 30 years after it was first proposed, through the 1920 and `30s which all seemed to have gaps in them. Until finally around 1950 two Russians mathematicians, one of whom is extremely well-known, Petrovski, a specialist in systems of ordinary differential equations published a long and difficult, complicated 100 page paper in which they proved that the maximum number is three.

I won't put down their names. Petrovski-Landis. The maximum number was three. And then not many people were able to read the paper, and those who did there seemed to be gaps in the reasoning in various places until finally Arnold who was the greatest Russian, in my opinion, one of the greatest Russian mathematician, certainly in this field of analysis and differential equations, but in other fields, too, he still is great, although he is somewhat older now, criticized it.

He said look, there is a really big gap in this argument and it really cannot be considered to be proven. People tried working very hard to patch it up and without success. Then about 1972 or so, '75 maybe, a Chinese mathematician found a system with four. Wrote down the numbers, the number a is so much, b is so much, and they were absurd numbers like 10^-6 and 40 billion and so on, nothing you could plot on a computer screen, but found a system with four. Nobody after this tried to fill in the gap in the Petrovski-Landis paper.

I was then chairman of the math department, and one of my tasks was protocol and so on. Anyway, we were trying very hard to attract a Chinese mathematician to our department to become a full professor. He was a really outstanding analyst and specialist in various fields. Anyway, he came in for a courtesy interview and we chatted. At the time, I was very much interested in limit cycles. And I had on my desk the Math Society's translation of the Chinese book on limit cycles.

A collection of papers by Chinese mathematicians all on limit cycles. After a certain point he said, oh, I see you're interested in limit cycle problems. I said yeah, in particular, I was reading this paper of the mathematician who found four limit cycles. And I opened to that system and said the name is, and I read it out loud. I said do you by any chance know him? And he smiled and said yes, very well. That is my mother. [LAUGHTER]

Well, bye-bye.