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Dynamics of the p -adic Shift and Applications

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DYNAMICS OF THE p-ADIC SHIFT AND APPLICATIONS

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Abstract. There is a natural continuous realization of the one-sided Bernoulli shift on the p-adic integers as the map that shifts the coefficients of the p-adic expansion to the left. We study this map's Mahler power series expansion. We prove strong results on p-adic valuations of the coefficients in this expansion, and show that certain natural maps (including many polynomials) are in a sense small perturbations of the shift. As a result, these polynomials share the shift map's important dynamical properties. This provides a novel approach to an earlier result of the authors.

1. **Introduction.** In recent years, several authors have studied the dynamics that result from various maps on the p-adics. In many cases they have shown that relatively simple and natural transformations satisfy important dynamical properties, such as ergodicity. For an overview of this work, the reader may refer to the recent monographs [1, 6, 11], the survey [3], and the references therein. General references are in [5, 9, 10].

In this paper, we continue this line of research by considering noninvertible Bernoulli transformations on the p-adic integers. Bernoulli transformations are those isomorphic with the "left shift" of infinite sequences on some alphabet. They are ubiquitous throughout the field of dynamics, and come up in numerous guises in several branches of mathematics. In measurable dynamics, the map $T_n : [0,1] \to [0,1]$ for n a positive integer, taking x to nx mod 1, provides a simple example of a (noninvertible) Bernoulli transformation. Taking base-n expansions of x, and ignoring the (measure zero) set of redundant expansions $\overline{.n-1}$, one sees that indeed this map is just a left shift.

Moving to the p-adic context, the aim of this paper is to present a novel way of realizing the one-sided Bernoulli shift on p symbols, where p is some prime. We do this by starting with the most natural realization: Any $x \in \mathbb{Z}_p$ has a unique (possibly infinite) expansion of the form $x = \sum_{i=0}^{\infty} b_i p^i$ ($b_i \in \{0, 1, \ldots, p-1\}$); one

can define the "p-adic shift" $\mathcal{S} : \mathbb{Z}_p \to \mathbb{Z}_p$ to be a left shift on this expansion, that is $\mathcal{S}(x) = \sum_{i=0}^{\infty} b_{i+1} p^i$. By showing that suitably small perturbations of $\mathcal{S}$ are still Bernoulli, we can find many "nice" maps, such as polynomials, that behave like the shift map $\mathcal{S}$ in this way. We originally discussed these maps in [7]; this paper presents a novel and more direct way of obtaining them, as we will remark below, and it can be read independently of [7]. Because we are working on the p-adics, our proofs use divisibility properties of certain polynomials, and thus, we obtain a connection between dynamics and number theory.

Our work was motivated by the article [3], which studied the measurable dynamics of polynomial maps on $\mathbb{Z}_p$. The authors in [3] asked when polynomial maps can satisfy the measurable dynamical property of being mixing. (It turns out that this was known to Woodcock and Smart, who showed in [13] that the polynomial map $x \mapsto \frac{x^p - x}{p}$ defines a Bernoulli, hence mixing, transformation on $\mathbb{Z}_p$.) In [7], the authors gave a detailed account of the dynamics that can result from a certain well-behaved class of maps on the p-adics. In particular, we introduced a set of conditions on the Mahler expansion of a transformation on the p-adics which are sufficient for it to be Bernoulli (see Definition 3.10). We call the class of maps meeting those conditions the *Mahler-Bernoulli class*.

The sufficiency of these conditions was proved in [7] via a structure theorem for so-called "locally scaling" transformations (see Definition 3.1). The Mahler-Bernoulli class conditions mentioned above were (roughly) that the transformation be a small enough perturbation of $x \mapsto \binom{x}{p}$ (which is part of the Mahler basis), and so applying the structure theorem required a careful study of the dynamics of the map $x \mapsto \binom{x}{p}$, including showing that it is Bernoulli. The approach of the present paper turns out to be more direct than that of [7], to which it is in a sense dual: we begin with the easily understood dynamics of $\mathcal{S}$ and work to better understand its Mahler expansion, proving that it is "well-approximated" by $x \mapsto \binom{x}{p}$. Since we are working in an ultrametric setting, the Mahler-Bernoulli class conditions may be reformulated as defining exactly those maps that are small enough perturbations of $\mathcal{S}$, giving a more conceptual reason why they should be Bernoulli!

This paper is organized as follows. We begin by studying the p-adic shift and its properties. Section 3 is devoted to developing our machinery and proving the main result. We introduce locally scaling transformations (similarly to our discussion in [7]) and then show what we mean when we say that certain maps are small perturbations of the shift map. At the end of the section, we prove that certain locally scaling transformations are indeed small perturbations of the shift map, and hence Bernoulli. In particular, we define the Mahler-Bernoulli class and show that all members thereof satisfy these properties. Finally, in Section 4 we briefly talk about maps related to the p-adic shift.

We have discussed locally scaling transformations and their dynamical properties in [7]. However, the interpretation that certain locally scaling transformations are in some sense small perturbation on the shift map, and therefore remain Bernoulli, is original to this paper (and is its main contribution).

The p-adic shift map has been previously investigated by other researchers. In particular, ergodicity of the p-adic shift was established in [8]. Conditions for the equivalence of a quadratic map to the shift were studied in [12], and a realization of the Smale horseshoe map as a shift on $\mathbb{Z}_p \times \mathbb{Z}_p$ was considered in [2].

2. **The p-adic shift.** Throughout the rest of this paper, we fix a prime p, and let $|\cdot|$ be the p-adic absolute value.

The p-adic shift $\mathcal{S}: \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$ is defined as follows. If $x = b_0 + b_1 p + b_2 p^2 + \cdots$, where the $b_i \in \{0, 1, \ldots, p-1\}$, we let $\mathcal{S}(x) = b_1 + b_2 p + b_3 p^2 + \cdots$. We immediately see that if $\mathcal{S}^k$ denotes the k-fold iterate of $\mathcal{S}$, then we have that $\mathcal{S}^k(x) = b_k + b_{k+1} p + \cdots$. Moreover, for $x \in \mathbb{Z}$, it is the case that $\mathcal{S}^k(x) = \lfloor x/p^k \rfloor$ where $\lfloor \cdot \rfloor$ is the greatest integer function.

Clearly, the shift is continuous on $\mathbb{Z}_p$ with respect to the p-adic metric (where the distance between x and y is $|x - y|$).

By Mahler's Theorem, any continuous $T: \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$ can be expressed in the form of a uniformly convergent series, called its *Mahler Expansion*:

$$T(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$$

where

$$a_n = \sum_{i=0}^{n} (-1)^{n+i} T(i) \binom{n}{i} \in \mathbb{Z}_p \tag{1}$$

Remark 2.1. For an arbitrary function $T: \mathbb{Z}_p \to \mathbb{Z}_p$, we can write down the formal identity $T(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$ for coefficients $a_n \in \mathbb{Z}_p$ to be determined. Then, substituting in $x = 0, 1, 2, \ldots$ in turn, we can inductively determine that the a_n would have to be as in (1), noting that only finitely many summands will be non-zero at each stage. The true content of Mahler's Theorem is that for T continuous on $\mathbb{Z}_p$, this series converges, which happens if and only if $|a_n| \to 0$ as $n \to \infty$ by convergence properties on the p-adics.

Our first goal is to study the Mahler expansion of $\mathcal{S}^k$. Throughout this paper, we let $a_n^{(k)}$ be the nth Mahler coefficient of $\mathcal{S}^k$. In other words, we have

$$\mathcal{S}^k(x) = \sum_{n=0}^{\infty} a_n^{(k)} \binom{x}{n}.$$

Theorem 2.2. *The coefficients $a_n^{(k)}$ satisfy the following properties:*

(i) $a_n^{(k)} = 0$ *for* $0 \le n < p^k$;
(ii) $a_n^{(k)} = 1$ *for* $n = p^k$;
(iii) Suppose $j \ge 0$. *Then,* p^j *divides* $a_n^{(k)}$ *for* $n > jp^k - j + 1$ *(and so,* $|a_n^{(k)}| \le 1/p^j$).

Proof. Our proof is closely based on Elkies's short derivation of Mahler's Theorem [4]. Let

$$F(t) = \sum_{n \ge 0} \mathcal{S}^k(n) t^n \in \mathbb{Z}_p[[t]] \qquad \text{and} \qquad A(u) = \sum_{n \ge 0} a_n^{(k)} u^n \in \mathbb{Z}_p[[u]].$$

Recall the standard power series identities

$$\frac{1}{1-t} = \sum_{i \ge 0} t^i \qquad \text{and} \qquad \frac{t}{(1-t)^2} = \sum_{i \ge 1} i t^i.$$

Using them and the definition of $\mathcal{S}^k$, we may compute that

$$
\begin{aligned}
F(t) &= \sum_{n \geq 0} \mathcal{S}^k(n) t^n = \sum_{a=0}^{p^k-1} \sum_{b \geq 0} b t^{a+bp^k} \\
&= \left(\sum_{a=0}^{p^k-1} t^a \right) \left(\sum_{b \geq 0} b (t^{p^k})^b \right) = \left(\frac{1-t^{p^k}}{1-t} \right) \left(\frac{t^{p^k}}{(1-t^{p^k})^2} \right) \\
&= \frac{t^{p^k}}{(1-t)(1-t^{p^k})}
\end{aligned}
$$

Before proceeding, we remark that

$$
A(u) = \frac{1}{1+u} F\left(\frac{u}{1+u} \right). \tag{2}
$$

Indeed, suppose $T : \mathbb{Z}_p \to \mathbb{Q}_p$ is any map. Set $\tilde{F}(t) = \sum_{n \geq 0} T(n) t^n$ and $\tilde{A}(u) = \sum_{n \geq 0} a_n u^n$ where a_n is such that $\sum_i a_i \binom{n}{i} = T(n)$ for all $k \geq 0$. Then,

$$
\begin{aligned}
\tilde{F}(t) &= \sum_{n \geq 0} \sum_{i=0}^{n} a_i \binom{n}{i} t^n = \sum_{i \geq 0} a_i \sum_{n \geq i} \binom{n}{i} t^n = \sum_{i \geq 0} a_i \frac{t^i}{(1-t)^{i+1}} \\
&= \left(\frac{1}{1-t} \right) \sum_{i \geq 0} a_i \left(\frac{t}{1-t} \right)^i = \frac{1}{1-t} \tilde{A}\left(\frac{t}{1-t} \right)
\end{aligned}
$$

From this, (2) follows by taking $\tilde{F} = F$, $\tilde{A} = A$, and $t = u/(1+u)$.

Now, note that because $p | \binom{p^k}{i}$ for all $0 < i < p^k$, we have $(1+u)^{p^k} - u^{p^k} = 1 + pR(u)$ where $R(u)$ is a polynomial of degree $p^k - 1$ in u without leading term, so that

$$
A(u) = \frac{1}{1+u} F\left(\frac{u}{1+u} \right) = \frac{u^{p^k}}{(1+u)^{p^k} - u^{p^k}},
$$

which is equal to $u^{p^k} \left(1 + pR(u) + p^2 R(u)^2 + p^3 R(u)^3 + \cdots \right)$. Since $R(u)$ has no leading term, we can now conclude (i) and (ii). The case $j = 0$ of (iii) is trivial, so we may assume $j \geq 1$. Working modulo p^j we obtain the equality (in $(\mathbb{Z}_p / p^j \mathbb{Z}_p)[[u]]$)

$$
A(u) \equiv u^{p^k} \left(1 + pR(u) + \cdots + p^{j-1} R(u)^{j-1} \right) \pmod{p^j}
$$

Note that the right hand side is a polynomial of degree $p^k + (j-1)(p^k - 1) = jp^k - j + 1$. This allows us to conclude (iii). $\qquad \square$

As we will see, the next corollary will be of fundamental importance in the following section, where we try to bound coefficients in the Mahler expansions of maps related to the shift.

Corollary 2.3. *The maximum possible value for $p^{\lfloor \log_p n \rfloor} |a_n^{(k)}|$ is p^k and it is attained only when $n = p^k$.*

Proof. We may assume $a_n^{(k)} \neq 0$. Let $v_p(a_n^{(k)})$ denote the integer ℓ so that p^ℓ precisely divides $a_n^{(k)}$. We are asked to prove that $\lfloor \log_p n \rfloor - v_p(a_n^{(k)}) \leq k$ with equality if and only if $n = p^k$. Setting $\ell = \lfloor \log_p n \rfloor - k$, we are thus to show that p^ℓ divides $a_n^{(k)}$ and $p^{\ell+1}$ divides it unless $n = p^k$.

Note that $p^\ell \geq \ell$ so that

$$n \geq p^{\lfloor \log_p n \rfloor} = p^\ell p^k \geq \ell p^k \geq \ell p^k - \ell + 1$$

and applying Theorem 2.2 yields that p^ℓ does in fact divide $a_n^{(k)}$.

If $\ell > 0$, then $p^\ell \geq \ell+1$ so that we in fact have $n \geq (\ell+1)p^k \geq (\ell+1)p^k - (\ell+1) + 1$ and so Theorem 2.2 yields that $p^{\ell+1}$ does in fact divide $a_n^{(k)}$. Since $a_n^{(k)} = 0$ for $n < p^k$, by Theorem 2.2, it suffices to prove that p divides $a_n^{(k)}$ for $n > p^k$; but this is immediate from Theorem 2.2. $\qquad\square$

3. **Perturbing the shift.**

3.1. **Local scaling.** The shift $\mathcal{S}$ cuts off the first digit term in the p-adic expansion of $x \in \mathbb{Z}_p$. Notice that if the expansions of x and y agree on the first digit (i.e. $|x - y| \leq 1/p$), then $\mathcal{S}$ multiplies the distance between them by p. Thus, $\mathcal{S}$ scales distances between points that are close enough.

This observation motivates the following definition from [7].

Definition 3.1. We say that $T : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$ is *locally scaling* with scaling radius r and scaling constant $C \geq 1$, and write that T is (r, C)-locally scaling, if for all x, y with $|x - y| \leq r$, we have $|T(x) - T(y)| = C|x - y|$. We will always assume, without loss of generality, that $r = p^\ell$ and $C = p^m$, where $\ell \leq 0$ and $m \geq 0$.

Proposition 3.2. *The map $\mathcal{S}^k$ is (p^{-k}, p^k)-locally scaling.*

Proof. Immediate. $\qquad\square$

Notice that if T is a locally scaling map, it is continuous, and for $r' \leq r$, the restriction $T\mid_{B_{r'}(x)}$ is injective into $B_{Cr'}(T(x))$. It is surjective as well, as the following lemma in [7] shows; the proof is included for completeness.

Proposition 3.3. *For T an (r, C)-locally scaling map and $r' \leq r$, the restricted map $T\mid_{B_{r'}(x)} : B_{r'}(x) \longrightarrow B_{Cr'}(T(x))$ is a bijection.*

Proof. Let $S = T\mid_{B_{r'}(x)}$. Because injectivity of S is clear, we just prove surjectivity. Let $B \subset B_{Cr'}(T(x))$ be a ball of radius $p^{-j} \leq Cr'$ for $j \in \mathbb{Z}_{\geq 0}$. Assume furthermore that there are η balls of radius $(1/C)\, p^{-j}$ contained in $B_{r'}(x)$. Pick one point in each of these balls of radius $(1/C)\, p^{-j}$ (the *representative* of that ball). Because these η representatives are all at least $(1/C)\, p^{-j}$ apart from one another and $(1/C)\, p^{-j} \leq r'$, their images have to be at least p^{-j} apart from one another by local scaling. Thus, they have to occupy η distinct balls of radius p^{-j} in the range. Finally, because the number of balls of radius p^{-j} contained in $B_{Cr'}(T(x))$ is also η, one of the representatives in $B_{r'}(x)$ must map into B by the pigeonhole principle.

We conclude that the image of S is dense in $B_{Cr'}(T(x))$. Then, since S is continuous and $B_{r'}(x)$ is compact, $S(B_{r'}(x))$ must be closed. Thus it is all of $B_{Cr'}(T(x))$. $\qquad\square$

3.2. (p^{-k}, p^k)-**locally scaling maps.** We now turn our attention to a special case of local scaling, the (p^{-k}, p^k)-locally scaling maps. We will see that they behave "nicely" under preimages. More precisely, when one takes the preimage of a given ball, the result is a union of smaller balls that are very evenly distributed throughout $\mathbb{Z}_p$.

We recall that for each $k \geq 1$, the set $\mathbb{Z}_p$ can be partitioned into p^k balls of radius p^{-k} and we shall refer to these as the *balls of radius p^{-k} in $\mathbb{Z}_p$*. Also, we denote
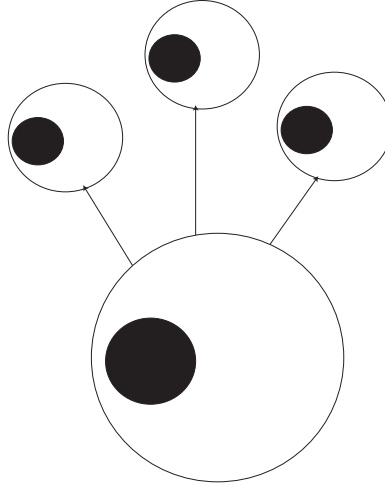
FIGURE 1.   An illustration of the local nesting property for $(1/3, 3)$-locally scaling maps. Let B be the big ball at the bottom of the figure; the arrows lead to its preimages B_1, B_2, and B_3. The shaded ball inside B is some B'. Its preimage consists of the balls B'_1, B'_2 and B'_3, one inside each of the B_i. The figure should not be taken to imply that B and the B_i are disjoint; in fact, if $B = \mathbb{Z}_p$, it will contain all the B_i.

Haar measure on $\mathbb{Z}_p$ by μ and recall that it is completely determined by its value on balls, the measure of each ball being its radius.

Let T be a (p^{-k}, p^k)-locally scaling map. In this case, each of the p^k balls of radius p^{-k} in $\mathbb{Z}_p$ is mapped bijectively onto $\mathbb{Z}_p$. It follows that given a ball $B \subset \mathbb{Z}_p$ of radius p^{-j}, with $j \geq k$, its preimage is

$$T^{-1}(B) = \bigsqcup_{i=1}^{p^k} B_i$$

where each B_i is a ball of radius $p^{-(j+k)}$, and the B_i are contained in distinct balls of radius p^{-k}. Furthermore, if $B' \subset B$ is a ball of radius $p^{-j'} \leq p^{-j}$, then

$$T^{-1}(B') = \bigsqcup_{i=1}^{p^k} B'_i$$

where each B'_i is a ball of radius $p^{-(j'+k)}$, and each B'_i is contained in B_i. We call this very nice property of preimages the *nesting property* (see Figure 1 for an illustration).

We have proved:

Lemma 3.4. *Let $T : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$ be (p^{-k}, p^k)-locally scaling. Suppose that B is a ball of radius p^{-kj}. Then, $T^{-1}(B)$ consists of a union of p^k balls of radius $p^{-k(j+1)}$, one inside each ball of radius p^{-k} in $\mathbb{Z}_p$. In particular, T is Haar measure-preserving.*

3.3. **Connections with Bernoulli maps.** With our setup, we will be able to say that certain maps (namely the (p^{-k}, p^k)-locally scaling ones) in many senses behave "the same" as the p-adic Bernoulli shift (or one of its iterates). To make these

notions more precise, we introduce some concepts in topological and measurable dynamics.

Definition 3.5. Two maps $T : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$ and $S : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$ are said to be *topologically isomorphic* if there exists a homeomorphism $\Phi : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$ such that

$$\Phi \circ T(x) = S \circ \Phi(x) \tag{3}$$

for all x.

The maps are *measurably isomorphic* if there exists an invertible and measure-preserving map Φ such that (3) holds for almost all x in $\mathbb{Z}_p$.

Close variants of the following theorem and proof are in [7].

Theorem 3.6. *Let $T : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$ be a (p^{-k}, p^k)-locally scaling map. Then T is topologically and measurably isomorphic to $\mathcal{S}^k$.*

Proof. We will find a measure-preserving homeomorphism $\Phi : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$ such that $\Phi \circ T = \mathcal{S}^k \circ \Phi$. Consider

$$\Phi(x) = \sum_{i=0}^{\infty} d_i(x) p^i,$$

where, if $i = qk + r$ for integer q and r with $0 \leq r < k$, we have that $d_i(x) = (T^q(x))_r$ (the rth digit in the p-adic expansion of $T^q(x)$). From this definition, it is easy to see that $\Phi \circ T(x) = \mathcal{S}^k \circ \Phi(x)$ for all $x \in \mathbb{Z}_p$.

We now proceed to show that Φ, as defined, is continuous, bijective and measure-preserving. Because $\mathbb{Z}_p$ is compact, continuity of the inverse follows by general facts in topology. Therefore, we will have proved that Φ is a homeomorphism.

To show that Φ is continuous, suppose that $|x - y| \leq p^{-k\eta}$ for $\eta \geq 1$. It follows by local scaling that $|T(x) - T(y)| \leq p^{-k(\eta-1)}$, and in general, we can see that for $q \leq \eta - 1$, we have that $|T^q(x) - T^q(y)| \leq p^{-k}$. Therefore, the p-adic expansions of $\Phi(x)$ and $\Phi(y)$ agree at least up to the $p^{-(k\eta-1)}$ term, and we see that $|\Phi(x) - \Phi(y)| \leq p^{-k\eta}$. Continuity follows.

To show injectivity, suppose that $\Phi(x) = \Phi(y)$. Then, $|T^i(x) - T^i(y)| \leq p^{-k}$ for all $i \geq 0$. But if $|T^i(x) - T^i(y)| \neq 0$ for some i, it follows by local scaling that there is a $j \geq 0$ such that $|T^{i+j}(x) - T^{i+j}(y)| > p^{-k}$, a contradiction.

Suppose that $y \in \mathbb{Z}_p$ and $\eta \geq 1$. From the definition of Φ, it is clear that

$$x \in \Phi^{-1}\left(B_{p^{-k\eta}}(y)\right)$$

if and only if

$$
\begin{aligned}
x &\in & B_{p^{-k}}(y) \\
T(x) &\in & B_{p^{-k}}\left(\mathcal{S}^k(y)\right) \\
&\vdots& \\
T^{\eta-1}(x) &\in & B_{p^{-k}}\left(\mathcal{S}^{(\eta-1)k}(y)\right).
\end{aligned}
$$

The balls of radius $p^{-k\eta}$ for $\eta \geq 1$ generate the Borel σ-algebra of $\mathbb{Z}_p$. So, in order to prove that Φ is measure-preserving it suffices to prove that

$$\mu\left(\Phi^{-1}\left(B_{p^{-k\eta}}(y)\right)\right) = \mu\left(B_{p^{-k\eta}}(y)\right) = p^{-k\eta}.$$

Since $\mathbb{Z}_p$ is compact, this will also imply that Φ is surjective since $\Phi^{-1}(y)$ will then be the descending intersection of nonempty closed sets.

We will prove by induction on $\eta \geq 0$ that $\Phi^{-1}\left(B_{p^{-k\eta}}(y)\right)$ is a single ball of radius $p^{-k\eta}$. The case $\eta = 0$ is immediate. If $\eta \geq 1$, then

$$
\begin{aligned}
\Phi^{-1}\left(B_{p^{-k\eta}}(y)\right) &= \bigcap_{i=0}^{\eta-1} T^{-i} B_{p^{-k}}\left(\mathcal{S}^{ik}(y)\right) \\
&= B_{p^{-k}}(y) \cap T^{-1}\left(\bigcap_{i=1}^{\eta-1} T^{-(i-1)} B_{p^{-k}}\left(\mathcal{S}^{ik}(y)\right)\right) \\
&= B_{p^{-k}}(y) \cap T^{-1}\Phi^{-1}\left(B_{p^{-k(\eta-1)}}\left(\mathcal{S}^k y\right)\right)
\end{aligned}
$$

is a ball of radius $p^{-k\eta}$ by the inductive hypothesis and Lemma 3.4. □

3.4. **"Small" perturbations on $\mathcal{S}^k$.** At this point in the presentation, we have determined that (p^{-k}, p^k)-locally scaling maps are Bernoulli. Furthermore, by understanding the Mahler expansion of the shift map $\mathcal{S}^k$, as we did earlier, we hope to understand the scaling properties of polynomial maps (namely, finite $\mathbb{Z}_p$-linear combinations of the $\binom{x}{n}$), and thus, to find Bernoulli polynomials. As we will see, we have an infinite class of polynomial maps such that any g in this class can be written as a sum of $\mathcal{S}^k$ and a perturbing factor satisfying the Lipschitz property (see below). In turn, this perturbing factor has small enough Lipschitz constant so that g is (p^{-k}, p^k)-locally scaling, like $\mathcal{S}^k$. We proceed to lay the groundwork for this argument.

Definition 3.7. A function $T : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$ is C-Lipschitz if $|T(x) - T(y)| \leq C|x-y|$ for all $x, y \in \mathbb{Z}_p$.

If T is C-Lipschitz and $a \in \mathbb{Q}_p$, it is clear that aT is $|a|C$-Lipschitz. Also, because of the strong triangle inequality, if T_i is C_i-Lipschitz, $\sum T_i$ is $(\sup_i\{C_i\})$-Lipschitz, provided this supremum exists (i.e., the C_i are bounded).

Lipshitz maps are important in our discussion because they provide us with a way of slightly modifying a locally scaling function such that the resulting map is still locally scaling. The following proposition demonstrates one such method.

Proposition 3.8. *Let* $T : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$ *be* (r, C)-*locally scaling,* $S : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$ *be* D-*Lipshitz with* $D < C$, *and suppose that* $u \in \mathbb{Z}_p^{\times} = \{\mathbb{Z}_p : |x| = 1\}$. *Then* $\tilde{T} = uT + S$ *is* (r, C)-*locally scaling.*

Proof. Take x and y with $|x - y| \leq r$. Then

$$
\begin{aligned}
|\tilde{T}(x) - \tilde{T}(y)| &= |(uT(x) + S(x)) - (uT(y) + S(y))| \\
&= |u(T(x) - T(y)) + (S(x) - S(y))|.
\end{aligned}
$$

But, $|u(T(x) - T(y))| = C|x-y| > D|x-y| \geq |S(x) - S(y)|$. Therefore, we see that $|\tilde{T}(x) - \tilde{T}(y)| = C|x-y|$, as desired. □

In particular, if $\tilde{f}(x) = \mathcal{S}^k(x) + b(x)$ where b is p^{k-1}-Lipshitz, then $\tilde{f}$ is (p^{-k}, p^k)-locally scaling, and hence Bernoulli. This condition gives us sufficient conditions on the Mahler expansion of a continuous map $T : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$ for the map to be isomorphic to $\mathcal{S}^k$ for some k.

Before stating the result, however, we need the following standard lemma:

Lemma 3.9. *The* $\mathbb{Z}_p$ *map* $x \mapsto \binom{x}{n}$ *is* $p^{\lfloor \log_p n \rfloor}$-*Lipshitz for all* n.

Proof. See [9, pg. 227]. □

Definition 3.10. Let $T(x)$ be given by the uniformly converging Mahler series

$$T(x) = \sum_{n=0}^{\infty} A_n \binom{x}{n}$$

and set $C = \max_n p^{\lfloor \log_p n \rfloor} |A_n|$. Suppose that the following conditions hold:

- There exists a unique n_0 which attains this maximum;
- $n_0 = p^k$ for some $k > 0$;
- $|A_{n_0}| = 1$ (thus, $C = n_0 = p^k$).

Then, we say that T is in the *Mahler-Bernoulli class*.

Theorem 3.11. *Let T be in the Mahler-Bernoulli class, with $C = p^k$ as in the definition. Then T is (p^{-k}, p^k)-locally scaling and hence isomorphic to $\mathcal{S}^k$.*

Proof. A variant of this theorem is in [7]. The proof that follows uses the Mahler expansion of $\mathcal{S}^k$, and is original to this paper.

The idea of the proof is that by the Mahler-Bernoulli class conditions, the $A_{p_k} \binom{x}{p^k}$ term will be the dominant one in determining the scaling behavior of T. Indeed, as we will see below, after multiplying by a unit as appropriate, T and $\mathcal{S}^k$ differ by a p^{k-1}-Lipshitz component; this follows by the Mahler-Bernoulli class conditions, as well as the tight control guaranteed by Lemma 3.9. The details of this argument are below.

Recall that for the map $\mathcal{S}^k$, we have that $a_{p^k}^{(k)} = 1$. Therefore, the fact that $|A_{p_k}| = 1$ in the Mahler expansion of T implies that there exists $u \in \mathbb{Z}_p^{\times}$ such that $u a_{p^k}^{(k)} = A_{p_k}$. Consider the general term $b_n = u a_n^{(k)} - A_n$. By choice of u, we have $b_{p^k} = 0$. Furthermore, the strong triangle inequality tells us that $|b_n| \leq \max\{|a_n^{(k)}|, A_n\}$. Regardless of what the maximum is for a particular $n \neq p^k$, it is the case that $|b_n| p^{\lfloor \log_p n \rfloor} < p^k$ by definition and Corollary 2.3. Therefore, using Lemma 3.9, we see that $b_n \binom{x}{n}$ is p^{k-1}-Lipshitz for all n (since $b_{p^k} = 0$, the claim is trivial for $n = p^k$).

Hence, $u \mathcal{S}^k(x) - T(x) = \sum_{n=0}^{\infty} b_n \binom{x}{n}$ is p^{k-1}-Lipshitz. This implies, by Proposition 3.8, that T is (p^{-k}, p^k)-locally scaling and the theorem follows. □

Remark 3.12. The reader may (rightly) point out that proving the local scaling properties of maps like $\binom{x}{p^k}$ should not be much harder than proving Lipschitz properties of the $\binom{x}{n}$, which we did not demonstrate in this paper, but rather quoted from a standard text. Indeed, the local scaling proof was done directly in [7]. Going through the Mahler expansion of the shift map provides us with not as much a new proof of the local scaling properties, but an interpretation of the maps in the Mahler-Bernoulli class, which satisfy them.

4. **Other related maps.** The p-adic shift is actually a special case of another natural class of maps. First off, define $g : \mathbb{Q}_p \longrightarrow \mathbb{Z}_p$ as follows. Given $x \in \mathbb{Q}_p$, we can express it uniquely as $x = \sum_{i=\ell}^{\infty} b_i p^i$ where $\ell \leq 0$ and $b_i \in \{0, 1, \ldots, p-1\}$ for all i.

With our setup notation, we let $g(x) = \sum_{i=0}^{\infty} b_i p^i$. In other words, g chops off the negative powers of p in the p-adic expansion of x.

Take $a \in \mathbb{Q}_p$. Define $f_a : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$ by $f_a(x) = g(ax)$ where g is as above. Notice that the p-adic shift is f_a where $a = 1/p$.

We will mostly be interested in f_a with $|a| > 1$. Now, $|a| > 1$ implies that $a = a'/p^k$ for $a' \in \mathbb{Z}_p^\times$ and $k > 0$. Therefore, for $x \in \mathbb{Z}_p$, $f_a(x) = g(a'x/p^k)$ where $a'x \in \mathbb{Z}_p$ and so, $f_a(x) = \mathcal{S}^k(a'x)$.

We now make a few remarks about the topological dynamics of the maps f_a.

For a with $|a| < 1$, we have that $f_a(\mathbb{Z}_p) \subsetneq \mathbb{Z}_p$, so f_a is not surjective, and cannot be isomorphic to the shift.

The situation when $|a| > 1$ is quite different.

Theorem 4.1. *For $|a| = p^k$, with $k > 0$, f_a is (p^{-k}, p^k)-locally scaling, and hence isomorphic to $\mathcal{S}^k$.*

Proof. We know that $f_a(x) = \mathcal{S}^k(a'x)$ where $a' \in \mathbb{Z}_p^\times$. Take x and y with $|x - y| \leq p^{-k}$. Then, $|a'x - a'y| = |x - y| \leq p^{-k}$. Therefore,

$$|\mathcal{S}^k(a'x) - \mathcal{S}^k(a'y)| = p^k|a'x - a'y| = p^k|x - y|$$

and so f_a is (p^{-k}, p^k)-locally scaling. $\square$

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